Sample Problems for the Third In-Class Exam
Math 246, Fall 2018, Professor David Levermore

(1) Compute the Laplace transform of $f(t) = t\,e^{3t}u(t-2)$ from its definition.

(2) Consider the following (old style) MATLAB commands.

```
>> syms t s Y; f = ['heaviside(t)*t^2 + heaviside(t − 3)*(3*t − t^2)'];
>> diffeqn = sym('D(D(y))(t) − 6*D(y)(t) + 10*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s)', 'y(0)', 'D(y)(0)'}, {Y, 2, 3});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) Give the initial-value problem for $y(t)$ that is being solved.
(b) Find the Laplace transform $Y(s)$ of the solution $y(t)$.

DO NOT take the inverse Laplace transform of $Y(s)$ to find $y(t)$, just solve for $Y(s)$! You may refer to the table on the last page.

(3) Find $Y(s) = \mathcal{L}[y](s)$ where $y(t)$ solves the initial-value problem

$$y'' + 4y' + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \leq t < 2\pi\,, \\ t - 2\pi & \text{for } t \geq 2\pi\,. \end{cases}$$

DO NOT take the inverse Laplace transform of $Y(s)$ to find $y(t)$, just solve for $Y(s)$! You may refer to the table on the last page.

(4) Find $X(s) = \mathcal{L}[x](s)$ where $x(t)$ solves the initial-value problem

$$x'' + 4x = \delta(t-3)\,, \qquad x(0) = 5\,, \quad x'(0) = 0\,.$$

DO NOT take the inverse Laplace transform of $X(s)$ to find $x(t)$, just solve for $X(s)$! You may refer to the table on the last page.

(5) Find the inverse Laplace transforms of the following functions.

(a) $F(s) = \dfrac{2}{(s+5)^2}\,,$

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}\,,$

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}\,.$

You may refer to the table on the last page.

(6) For each of the following differential operators compute its Green function $g(t)$ and its natural fundamental set for $t = 0$.

(a) $\mathrm{L} = \mathrm{D}^4 + 8\mathrm{D}^2 - 9$,

(b) $\mathrm{L} = (\mathrm{D} - 2)^3$.

You may refer to the table on the last page.

(7) Recast the equation $u''' + t^2u' - 3u = \sinh(2t)$ as a first-order system of ordinary differential equations.

(8) Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 45 liters and the second contains 30 liters. Brine with a salt concentration of 5 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 8 liters per hour, from the second into the first at 7 liters per hour, from the first into a drain at 4 liter per hour, and from the second into a drain at 3 liters per hour. At $t = 0$ there are 27 grams of salt in the first tank and 18 grams in the second.

(a) Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(b) Give the interval of definition for the solution of this initial-value problem.

(9) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

(a) $\mathbf{A}^{\mathrm{T}}$,

(b) $\overline{\mathbf{A}}$,

(c) $\mathbf{A}^{\mathrm{H}}$,

(d) $5\mathbf{A} - \mathbf{B}$,

(e) $\mathbf{AB}$,

(f) $\mathbf{B}^{-1}$.

(10) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x},$$

wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) Give a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b).

(d) For the system found in part (b), solve the initial-value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}, \qquad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(e) For the $\mathbf{A}(t)$ found in part (b), give the Green matrix for the system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t).$$

(11) Compute $e^{t\mathbf{A}}$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

(12) Give the Green matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$ when

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

(13) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix}.$$

Compute $e^{t\mathbf{A}}$ given that the characteristic polynomial of $\mathbf{A}$ is $p(z) = z^3 + 9z$ and that the natural fundamental set of solutions associated with $t = 0$ for $\mathrm{D}^3 + 9\mathrm{D}$ is

$$N_0(t) = 1\,, \qquad N_1(t) = \tfrac{1}{3}\sin(3t)\,, \qquad N_2(t) = \tfrac{1}{9}\big(1 - \cos(3t)\big)\,.$$

(14) Solve each of the following initial-value problems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

(15) Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(16) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 1.

(17) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}.$$

(a) Find all the eigenvalues of $\mathbf{A}$.
(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.
(c) Diagonalize $\mathbf{A}$.
(d) Compute $e^{t\mathbf{A}}$.
(e) Compute $(s\mathbf{I} - \mathbf{A})^{-1}$ for every s where it is defined.

(18) What answer will be produced by the following Matlab command?

```
>> A = [1 4; 3 2]; [vect, val] = eig(sym(A))
```

You do not have to give the answer in Matlab format.

(19) A 3×3 matrix $\mathbf{A}$ has the eigenpairs

$$\left(-3\,, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \qquad \left(2\,, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right), \qquad \left(5\,, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right).$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{A}}$!)
(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[j'(t)](s) = sJ(s) - j(0) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s),\ c \geq 0,$$

and u is the unit step function.

$$\mathcal{L}[\delta(t-c)h(t)](s) = e^{-cs}h(c) \qquad \text{where } c \geq 0 \text{ and } \delta \text{ is the unit impluse.}$$