Third In-Class Exam Solutions
Math 246, Professor David Levermore
Thursday, 15 November 2018

(1) [6] Recast the ordinary differential equation $v'''' - v^2 v''' + \cos(v'') + e^v v' - t^4 = 0$ as a first-order system of ordinary differential equations.

Solution. The normal form of the equation is

$$v'''' = v^2 v''' - \cos(v'') - e^v v' + t^4 .$$

Because this equation is fourth order, the first-order system must have dimension at least four. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_1^2 x_4 - \cos(x_3) - e^{x_1} x_2 + t^4 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} v \\ v' \\ v'' \\ v''' \end{pmatrix}.$$

(2) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 4+t^6 \\ -t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} 3t^4 \\ 1 \end{pmatrix}$.

(a) [2] Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.
(b) [3] Find $\mathbf{B}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{B}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.
(c) [2] Give a general solution to the system found in part (b).
(d) [3] Compute the Green matrix associated with the system found in part (b).

Solution (a). The Wronskian is

$$\begin{aligned}\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 4+t^6 & 3t^4 \\ -t^2 & 1 \end{pmatrix} &= (4+t^6)\cdot 1 - (-t^2)\cdot(3t^4) \\ &= 4 + 4t^6 = 4(1+t^6) .\end{aligned}$$

Solution (b). If $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions for the system $\mathbf{x}' = \mathbf{B}(t)\mathbf{x}$ then a fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{pmatrix} 4+t^6 & 3t^4 \\ -t^2 & 1 \end{pmatrix} .$$

Because any fundamental matrix satisfies $\mathbf{\Psi}'(t) = \mathbf{B}(t)\mathbf{\Psi}(t)$, we see that

$$\begin{aligned}\mathbf{B}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 6t^5 & 12t^3 \\ -2t & 0 \end{pmatrix}\begin{pmatrix} 4+t^6 & 3t^4 \\ -t^2 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{4(1+t^6)}\begin{pmatrix} 6t^5 & 12t^3 \\ -2t & 0 \end{pmatrix}\begin{pmatrix} 1 & -3t^4 \\ t^2 & 4+t^6 \end{pmatrix} \\ &= \frac{1}{4(1+t^6)}\begin{pmatrix} 18t^5 & 48t^3 - 6t^9 \\ -2t & 6t^5 \end{pmatrix} .\end{aligned}$$

Solution (c). A general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} 4+t^6 \\ -t^2 \end{pmatrix} + c_2\begin{pmatrix} 3t^4 \\ 1 \end{pmatrix} .$$

Solution (d). By using the fundamental matrix $\mathbf{\Psi}(t)$ from part (b) we find that the Green matrix is

$$\begin{aligned}\mathbf{G}(t,s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} &= \begin{pmatrix} 4+t^6 & 3t^4 \\ -t^2 & 1 \end{pmatrix} \begin{pmatrix} 4+s^6 & 3s^4 \\ -s^2 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{4(1+s^6)} \begin{pmatrix} 4+t^6 & 3t^4 \\ -t^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3s^4 \\ s^2 & 4+s^6 \end{pmatrix} \\ &= \frac{1}{4(1+s^6)} \begin{pmatrix} 4+t^6+3t^4s^2 & 12(t^4-s^4)+3t^4s^4(s^2-t^2) \\ s^2-t^2 & 4+s^6+3t^2s^4 \end{pmatrix}.\end{aligned}$$

(3) [6] Given that -3 is an eigenvalue of the matrix

$$\mathbf{C} = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & -2 \\ 1 & 0 & 1 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{C}$ associated with -3.

Solution. The eigenvectors of $\mathbf{C}$ associated with -3 are all nonzero vectors $\mathbf{v}$ such that $\mathbf{C}\mathbf{v} = -3\mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ such that $(\mathbf{C}+3\mathbf{I})\mathbf{v} = \mathbf{0}$, which is

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & 1 & -2 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} 2v_1 + 3v_2 + 2v_3 &= 0, \\ v_2 - 2v_3 &= 0, \\ v_1 \qquad + 4v_3 &= 0. \end{aligned}$$

This system may be solved either by elimination or by row reduction. By either method its general solution is found to be

$$v_1 = -4\alpha, \quad v_2 = 2\alpha, \quad v_3 = \alpha, \qquad \text{for any constant } \alpha.$$

Therefore the eigenvectors of $\mathbf{B}$ associated with 2 each have the form

$$\alpha \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} \qquad \text{for some constant } \alpha \neq 0.$$

(4) [10] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ 5 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -4 & -2 \\ 5 & -6 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 10z + 34 = (z+5)^2 + 3^2.$$

This is a sum of squares with $\mu = -5$ and $\nu = 3$. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-5t}\left[\cos(3t)\mathbf{I} + \frac{\sin(3t)}{3}\left(\mathbf{A} - (-5)\mathbf{I}\right)\right] \\ &= e^{-5t}\left[\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(3t)}{3}\begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}\right] \\ &= e^{-5t}\begin{pmatrix} \cos(3t) + \frac{1}{3}\sin(3t) & -\frac{2}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) & \cos(3t) - \frac{1}{3}\sin(3t) \end{pmatrix}. \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A} + 5\mathbf{I}) = 0$!) Therefore the solution of the initial-value problem is

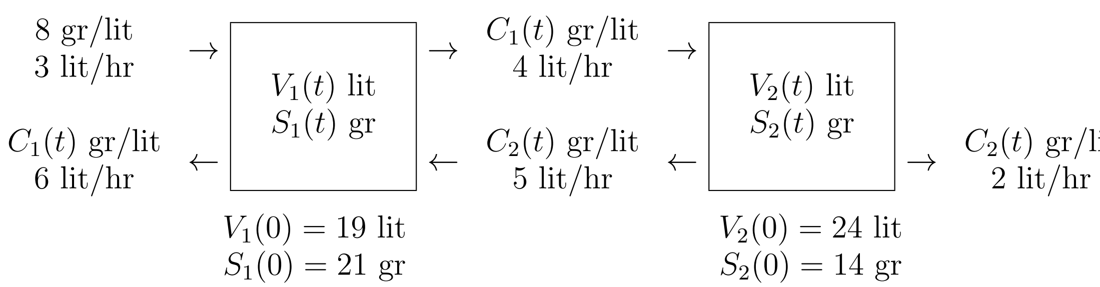
$$\begin{aligned} \mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^I &= e^{-5t}\begin{pmatrix} \cos(3t) + \frac{1}{3}\sin(3t) & -\frac{2}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) & \cos(3t) - \frac{1}{3}\sin(3t) \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ &= e^{-5t}\begin{pmatrix} -\frac{4}{3}\sin(3t) \\ 2\cos(3t) - \frac{2}{3}\sin(3t) \end{pmatrix}. \end{aligned}$$

(5) [8] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 19 liters and the second contains 24 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 3 liters per hour. Well-stirred brine flows from the first tank into the second at 4 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 6 liter per hour, and from the second into a drain at 2 liters per hour. At $t = 0$ there are 21 grams of salt in the first tank and 14 grams in the second.

(a) [6] Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(b) [2] Give the interval of definition for the solution of this initial-value problem.

Solution (a). Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t hours. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t hours. Because the mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.



We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will be $V_1(t) = 19 - 2t$ liters of brine in the first tank and $V_2(t) = 24 - 3t$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = 8\cdot 3 + \frac{S_2}{24-3t}\,5 - \frac{S_1}{19-2t}\,4 - \frac{S_1}{19-2t}\,6\,, \qquad S_1(0) = 21\,,$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{S_1}{19-2t}\,4 - \frac{S_2}{24-3t}\,5 - \frac{S_2}{24-3t}\,2\,, \qquad S_2(0) = 14\,.$$

Your answer could be left in the above form. However, it can be simplified to

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = 24 + \frac{5}{24-3t}\,S_2 - \frac{10}{19-2t}\,S_1\,, \qquad S_1(0) = 21\,,$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{4}{19-2t}\,S_1 - \frac{7}{24-3t}\,S_2\,, \qquad S_2(0) = 14\,.$$

Solution (b). This first-order system of differential equations is *linear*. Its coefficients are undefined either at $t = 8$ or $t = \frac{19}{2}$ and are continuous elsewhere. Its forcing is constant, so is continuous everywhere. Therefore the natural interval of definition for the solution of this initial-value problem is $(-\infty, 8)$ because:

- the initial time $t = 0$ is in $(-\infty, 8)$;
- all the coefficients and the forcing are continuous over $(-\infty, 8)$;
- two coefficients are undefined at $t = 8$.

However, it could also be argued that the interval of definition for the solution of this initial-value problem is $[0, 8)$ because the word problem starts at $t = 0$.

(6) [8] A 3×3 matrix $\mathbf{H}$ has the eigenpairs

$$\left(-3\,, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)\,, \qquad \left(-1\,, \begin{pmatrix}1\\0\\-1\end{pmatrix}\right)\,, \qquad \left(2\,, \begin{pmatrix}1\\-2\\1\end{pmatrix}\right)\,.$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{H}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{H}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{H}\mathbf{x}$.

Solution (a). One choice for $\mathbf{V}$ and $\mathbf{D}$ is

$$\mathbf{V} = \begin{pmatrix}1 & 1 & 1\\ 1 & 0 & -2\\ 1 & -1 & 1\end{pmatrix}\,, \qquad \mathbf{D} = \begin{pmatrix}-3 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{pmatrix}\,.$$

Solution (b). Use the given eigenpairs to construct the real eigensolutions

$$\mathbf{x}_1(t) = e^{-3t}\begin{pmatrix}1\\1\\1\end{pmatrix}\,, \qquad \mathbf{x}_2(t) = e^{-t}\begin{pmatrix}1\\0\\-1\end{pmatrix}\,, \qquad \mathbf{x}_3(t) = e^{2t}\begin{pmatrix}1\\-2\\1\end{pmatrix}\,.$$

Then a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \begin{pmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t)\end{pmatrix} = \begin{pmatrix}e^{-3t} & e^{-t} & e^{2t}\\ e^{-3t} & 0 & -2e^{2t}\\ e^{-3t} & -e^{-t} & e^{2t}\end{pmatrix}\,.$$

Alternative Solution (b). Given the $\mathbf{V}$ and $\mathbf{D}$ from part (a), a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} = \begin{pmatrix} e^{-3t} & e^{-t} & e^{2t} \\ e^{-3t} & 0 & -2e^{2t} \\ e^{-3t} & -e^{-t} & e^{2t} \end{pmatrix} .$$

(7) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

Solution by Eigen Methods. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z - 7 = (z-7)(z+1) .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 7 and -1. Consider the matrices

$$\mathbf{A} - 7\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 3 & -6 \end{pmatrix} , \qquad \mathbf{A} + 1\mathbf{I} = \begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix} .$$

After checking that the determinant of each matrix is zero, we can read off that eigenpairs of $\mathbf{A}$ are

$$\left(7 , \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) , \qquad \left(-1 , \begin{pmatrix} -2 \\ 3 \end{pmatrix}\right) .$$

Therefore a real general solution of the system is

$$\mathbf{x}(t) = c_1 e^{7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -2 \\ 3 \end{pmatrix} .$$

Solution by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z - 7 = (z-3)^2 - 9 - 7 = (z-3)^2 - 4^2 .$$

This is a difference of squares with $\mu = 3$ and $\nu = 4$. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t} \left[\cosh(4t)\mathbf{I} + \frac{\sinh(4t)}{4} \big(\mathbf{A} - 3\mathbf{I}\big) \right] \\ &= e^{3t} \left[\cosh(4t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(4t)}{4} \begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix} \right] \\ &= e^{3t} \begin{pmatrix} \cosh(4t) + \frac{1}{2}\sinh(4t) & \sinh(4t) \\ \frac{3}{4}\sinh(4t) & \cosh(4t) - \frac{1}{2}\sinh(4t) \end{pmatrix} . \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A}-3\mathbf{I})=0$!) Therefore a real general solution of the system is

$$\begin{aligned}\mathbf{x}(t)=e^{t\mathbf{A}}\mathbf{c}&=e^{3t}\begin{pmatrix}\cosh(4t)+\frac{1}{2}\sinh(4t) & \sinh(4t)\\ \frac{3}{4}\sinh(4t) & \cosh(4t)-\frac{1}{2}\sinh(4t)\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}\\ &=c_1e^{3t}\begin{pmatrix}\cosh(4t)+\frac{1}{2}\sinh(4t)\\ \frac{3}{4}\sinh(4t)\end{pmatrix}+c_2e^{3t}\begin{pmatrix}\sinh(4t)\\ \cosh(4t)-\frac{1}{2}\sinh(4t)\end{pmatrix}.\end{aligned}$$

(8) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-1 & -1\\ 1 & -3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A}=\begin{pmatrix}-1 & -1\\ 1 & -3\end{pmatrix}$ is

$$p(z)=z^2-\mathrm{tr}(\mathbf{A})z+\det(\mathbf{A})=z^2+4z+4=(z+2)^2.$$

This is a perfect square with $\mu=-2$. Then

$$\begin{aligned}e^{t\mathbf{A}}&=e^{-2t}\left[\mathbf{I}+t\left(\mathbf{A}-(-2)\mathbf{I}\right)\right]\\ &=e^{-2t}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+t\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\right]=e^{-2t}\begin{pmatrix}1+t & -t\\ t & 1-t\end{pmatrix}.\end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A}+2\mathbf{I})=0$!) Therefore a real general solution of the system is

$$\begin{aligned}\mathbf{x}(t)=e^{t\mathbf{A}}\mathbf{c}=e^{-2t}\begin{pmatrix}1+t & -t\\ t & 1-t\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}\\ =c_1e^{-2t}\begin{pmatrix}1+t\\ t\end{pmatrix}+c_2e^{-2t}\begin{pmatrix}-t\\ 1-t\end{pmatrix}.\end{aligned}$$

(9) [10] Find the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4+10\mathrm{D}^2+9$.

Solution from Green Function. The operator $\mathrm{D}^4+10\mathrm{D}^2+9$ has characteristic polynomial

$$p(s)=s^4+10s^2+9=(s^2+1)(s^2+9).$$

We have the partial-fraction identity

$$\frac{1}{p(s)}=\frac{1}{(s^2+1)(s^2+9)}=\frac{\frac{1}{8}}{s^2+1}+\frac{-\frac{1}{8}}{s^2+9}.$$

Referring to the table on the last page, item 2 with $a=0$ and $b=1$ and with $a=0$ and $b=3$ shows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t)=\sin(t),\qquad \mathcal{L}^{-1}\left[\frac{3}{s^2+3^2}\right](t)=\sin(3t).$$

Therefore the Green function for the operator $\mathrm{D}^4 + 10\mathrm{D}^2 + 9$ is

$$\begin{aligned} g(t) = \mathcal{L}^{-1}\bigg[\frac{1}{p(s)}\bigg](t) &= \tfrac{1}{8}\mathcal{L}^{-1}\bigg[\frac{1}{s^2+1}\bigg](t) - \tfrac{1}{24}\mathcal{L}^{-1}\bigg[\frac{3}{s^2+3^2}\bigg](t) \\ &= \tfrac{1}{8}\sin(t) - \tfrac{1}{24}\sin(3t)\,. \end{aligned}$$

Because we see the characteristic polynomial as

$$p(s) = s^4 + 0\cdot s^3 + 10\cdot s^2 + 0\cdot s + 9\,,$$

the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4 + 10\mathrm{D}^2 + 9$ is given by

$$\begin{aligned} N_3(t) &= g(t) &&= \tfrac{1}{8}\sin(t) - \tfrac{1}{24}\sin(3t)\,, \\ N_2(t) &= N_3'(t) + 0\cdot g(t) &&= \tfrac{1}{8}\cos(t) - \tfrac{1}{8}\cos(3t)\,, \\ N_1(t) &= N_2'(t) + 10\cdot g(t) &&= -\tfrac{1}{8}\sin(t) + \tfrac{3}{8}\sin(3t) + 10\big(\tfrac{1}{8}\sin(t) - \tfrac{1}{24}\sin(3t)\big)\,, \\ & &&= \tfrac{9}{8}\sin(t) - \tfrac{1}{24}\sin(3t)\,, \\ N_0(t) &= N_1'(t) + 0\cdot g(t) &&= \tfrac{9}{8}\cos(t) - \tfrac{1}{8}\cos(3t)\,. \end{aligned}$$

Solution from General Initial-Value Problem. For the operator $\mathrm{D}^4 + 10\mathrm{D}^2 + 9$ the general initial-value problem for initial-time 0 is

$$y'''' + 10y'' + y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \quad y''(0) = y_2\,, \quad y'''(0) = y_3\,.$$

Its characteristic polynomial is

$$p(z) = z^4 + 10z^2 + 9 = (z^2+1)(z^2+9) = (z^2+1)(z^2+3^2)\,,$$

which has roots i, $-i$, $i3$ and $-i3$. Therefore a real general solution is

$$y(t) = c_1\cos(t) + c_2\sin(t) + c_3\cos(3t) + c_4\sin(3t)\,.$$

Because

$$\begin{aligned} y'(t) &= -c_1\sin(t) + c_2\cos(t) - 3c_3\sin(3t) + 3c_4\cos(3t)\,, \\ y''(t) &= -c_1\cos(t) - c_2\sin(t) - 9c_3\cos(3t) - 9c_4\sin(3t)\,, \\ y'''(t) &= c_1\sin(t) - c_2\cos(t) + 27c_3\sin(3t) - 27c_4\cos(3t)\,, \end{aligned}$$

the general initial conditions yield the linear algebraic system

$$\begin{aligned} y_0 &= y(0) = c_1\cos(0) + c_2\sin(0) + c_3\cos(0) + c_4\sin(0) = c_1 + c_3\,. \\ y_1 &= y'(0) = -c_1\sin(0) + c_2\cos(0) - 3c_3\sin(0) + 3c_4\cos(0) = c_2 + 3c_4\,, \\ y_2 &= y''(0) = -c_1\cos(0) - c_2\sin(0) - 9c_3\cos(0) - 9c_4\sin(0) = -c_1 - 9c_3\,, \\ y_3 &= y'''(t) = c_1\sin(0) - c_2\cos(0) + 27c_3\sin(0) - 27c_4\cos(0) = -c_2 - 27c_4\,. \end{aligned}$$

This decouples into the two systems

$$\begin{aligned} y_0 &= c_1 + c_3\,, &\qquad y_1 &= c_2 + 3c_4\,, \\ y_2 &= -c_1 - 9c_3\,, &\qquad y_3 &= -c_2 - 27c_4\,. \end{aligned}$$

The solutions of these systems are

$$c_1 = \frac{9y_0 + y_2}{8}, \qquad c_2 = \frac{9y_1 + y_3}{8},$$
$$c_3 = -\frac{y_0 + y_2}{8}, \qquad c_4 = -\frac{y_1 + y_3}{24}.$$

Therefore the solution of the general initial-value problem is

$$\begin{aligned} y &= \frac{9y_0 + y_2}{8}\cos(t) + \frac{9y_1 + y_3}{8}\sin(t) - \frac{y_0 + y_2}{8}\cos(3t) - \frac{y_1 + y_3}{24}\sin(3t) \\ &= y_0\big(\tfrac{9}{8}\cos(t) - \tfrac{1}{8}\cos(3t)\big) + y_1\big(\tfrac{9}{8}\sin(t) - \tfrac{1}{24}\sin(3t)\big) \\ &\quad + y_2\big(\tfrac{1}{8}\cos(t) - \tfrac{1}{8}\cos(3t)\big) + y_3\big(\tfrac{1}{8}\sin(t) - \tfrac{1}{24}\sin(3t)\big)\,. \end{aligned}$$

We can read off from this that the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4 + 10\mathrm{D}^2 + 9$ is

$$N_0(t) = \tfrac{9}{8}\cos(t) - \tfrac{1}{8}\cos(3t)\,, \qquad N_1(t) = \tfrac{9}{8}\sin(t) - \tfrac{1}{24}\sin(3t)\,,$$
$$N_2(t) = \tfrac{1}{8}\cos(t) - \tfrac{1}{8}\cos(3t)\,, \qquad N_3(t) = \tfrac{1}{8}\sin(t) - \tfrac{1}{24}\sin(3t)\,.$$

(10) [8] Compute the Laplace transform of $f(t) = u(t-3)\,e^{-2t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\int_0^T e^{-st}u(t-3)\,e^{-2t}\,\mathrm{d}t = \lim_{T\to\infty}\int_3^T e^{-(s+2)t}\,\mathrm{d}t\,.$$

When $s \leq -2$ this limit diverges to $+\infty$ because in that case we have for every $T > 3$

$$\int_3^T e^{-(s+2)t}\,\mathrm{d}t \geq \int_3^T \mathrm{d}t = T - 3\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

When $s > -2$ we have for every $T > 3$

$$\int_3^T e^{-(s+2)t}\,\mathrm{d}t = -\frac{e^{-(s+2)t}}{s+2}\bigg|_3^T = -\frac{e^{-(s+2)T}}{s+2} + \frac{e^{-(s+2)3}}{s+2}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[-\frac{e^{-(s+2)T}}{s+2} + \frac{e^{-(s+2)3}}{s+2}\right] = \frac{e^{-(s+2)3}}{s+2} \qquad \text{for } s > -2\,.$$

Therefore the definition of the Laplace transform shows that

$$\mathcal{L}[f](s) = \begin{cases} \dfrac{e^{-(s+2)3}}{s+2} & \text{for } s > -2\,, \\ \text{undefined} & \text{for } s \leq -2\,. \end{cases}$$

(11) [10] Consider the following (old style) MATLAB commands.

```
>> syms t s X; f = ['t^2 + heaviside(t − 3)*(6 − t^2) − heaviside(t − 5)*6'];
>> diffeqn = sym('D(D(x))(t) + 2*D(x)(t) + 10*x(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(x(t),t,s),t,s)', 'x(0)', 'D(x)(0)'}, {X, 3, −7});
>> xtrans = simplify(solve(algeqn, X));
>> x = ilaplace(xtrans, s, t)
```

(a) [2] Give the initial-value problem for $x(t)$ that is being solved.
(b) [8] Find the Laplace transform $X(s)$ of the solution $x(t)$. (DO NOT take the inverse Laplace transform of $X(s)$ to find $x(t)$, just solve for $X(s)$!)

You may refer to the table on the last page.

Solution (a). The initial-value problem for $y(t)$ that is being solved is

$$x'' + 2x' + 10x = f(t)\,, \qquad x(0) = 3\,, \quad x'(0) = -7\,,$$

where the forcing $f(t)$ can be expressed either as the piecewise-defined function

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 3\,, \\ 6 & \text{for } 2 \le t < 5\,, \\ 0 & \text{for } 5 \le t\,, \end{cases}$$

or in terms of the unit step function as

$$f(t) = t^2 + u(t-3)(6-t^2) - u(t-5)6\,.$$

Solution (b). The Laplace transform of the initial-value problem is

$$\mathcal{L}[x''](s) + 2\mathcal{L}[x'](s) + 10\mathcal{L}[x](s) = \mathcal{L}[f](s)\,.$$

Because

$$\begin{aligned} \mathcal{L}[x](s) &= X(s)\,, \\ \mathcal{L}[x'](s) &= s\,\mathcal{L}[x](s) - x(0) = s\,X(s) - 3\,, \\ \mathcal{L}[x''](s) &= s\,\mathcal{L}[x'](s) - x'(0) = s^2X(s) - 3s + 7\,, \end{aligned}$$

the Laplace transform of the initial-value problem becomes

$$\big(s^2X(s) - 3s + 7\big) + 2\big(sX(s) - 3\big) + 10X(s) = \mathcal{L}[f](s)\,.$$

This simplifies to

$$(s^2 + 2s + 10)X(s) - 3s + 1 = \mathcal{L}[f](s)\,,$$

whereby

$$X(s) = \frac{1}{s^2 + 2s + 10}\big(3s - 1 + \mathcal{L}[f](s)\big)\,.$$

To compute $\mathcal{L}[f](s)$, we write $f(t)$ as

$$\begin{aligned} f(t) &= t^2 + u(t-3)(6-t^2) - u(t-5)6 \\ &= t^2 + u(t-3)j_1(t-3) + u(t-5)j_2(t-5)\,, \end{aligned}$$

where by setting $j_1(t-3) = 6 - t^2$ and $j_2(t-5) = -6$ we see by the shifty step method that

$$j_1(t) = 6 - (t+3)^2 = 6 - t^2 - 6t - 9 = -t^2 - 6t - 3\,, \qquad j_2(t) = -6\,.$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 0$, with $a = 0$ and $n = 1$, and with $a = 0$ and $n = 2$ shows that

$$\mathcal{L}[1](s) = \frac{1}{s}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,, \qquad \mathcal{L}[t^2](s) = \frac{2}{s^3}\,,$$

whereby item 6 with $c = 3$ and $j(t) = j_1(t)$ and with $c = 5$ and $j(t) = j_2(t)$ shows that

$$\begin{aligned} \mathcal{L}\big[u(t-3)j_1(t-3)\big](s) &= e^{-3s}\mathcal{L}[j_1](s) = -e^{-3s}\mathcal{L}[t^2 + 6t + 3](s) \\ &= -e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{3}{s}\right)\,, \\ \mathcal{L}\big[u(t-5)j_2(t-5)\big](s) &= e^{-5s}\mathcal{L}[j_2](s) = -e^{-5s}\mathcal{L}[6](s) = -e^{-5s}\frac{6}{s}\,. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}\big[t^2 + u(t-3)j_1(t-3) + u(t-5)j_2(t-5)\big](s) \\ &= \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{3}{s}\right) - e^{-5s}\frac{6}{s}\,. \end{aligned}$$

Upon placing this result into the expression for $Y(s)$ found earlier, we obtain

$$X(s) = \frac{1}{s^2 + 2s + 10}\left(3s - 1 + \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{3}{s}\right) - e^{-5s}\frac{6}{s}\right)\,.$$

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[Y(s)](t)$ of the function

$$Y(s) = e^{-3s}\,\frac{s-6}{s^2 + 4s + 20}\,.$$

You may refer to the table on the last page.

Solution. Referring to the table on the last page, item 6 with $c = 3$ implies that

$$\mathcal{L}^{-1}\big[e^{-3s}\,J(s)\big] = u(t-3)j(t-3)\,, \qquad \text{where} \qquad j(t) = \mathcal{L}^{-1}[J(s)](t)\,.$$

We apply this formula to

$$J(s) = \frac{s-6}{s^2 + 4s + 20}\,.$$

Because $s^2+4s+20=(s+2)^2+4^2$, we have the partial fraction identity

$$J(s)=\frac{s-6}{s^2+4s+20}=\frac{(s+2)-8}{(s+2)^2+4^2}=\frac{s+2}{(s+2)^2+4^2}-\frac{8}{(s+2)^2+4^2}\,.$$

Referring to the table on the last page, items 2 and 3 with $a=-2$ and $b=4$ imply that

$$\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+4^2}\right]=e^{-2t}\cos(4t)\,,\qquad \mathcal{L}^{-1}\left[\frac{4}{(s+2)^2+4^2}\right]=e^{-2t}\sin(4t)\,.$$

The above formulas and the linearity of the inverse Laplace transform yield

$$\begin{aligned}
j(t)=\mathcal{L}^{-1}[J(s)](t)&=\mathcal{L}^{-1}\left[\frac{s-6}{s^2+4s+20}\right](t)\\
&=\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+4^2}-\frac{8}{(s+2)^2+4^2}\right](t)\\
&=\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+4^2}\right](t)-2\mathcal{L}^{-1}\left[\frac{4}{(s+2)^2+4^2}\right](t)\\
&=e^{-2t}\cos(4t)-2e^{-2t}\sin(4t)\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{L}^{-1}\big[Y(s)\big](t)&=\mathcal{L}^{-1}[e^{-3s}J(s)](t)\\
&=u(t-3)j(t-3)\\
&=u(t-3)\Big(e^{-2(t-3)}\cos\big(4(t-3)\big)-2e^{-2(t-3)}\sin\big(4(t-3)\big)\Big)\,.
\end{aligned}$$

Table of Laplace Transforms

$\mathcal{L}[t^n e^{at}](s)=\dfrac{n!}{(s-a)^{n+1}}$	for $s>a$.
$\mathcal{L}[e^{at}\cos(bt)](s)=\dfrac{s-a}{(s-a)^2+b^2}$	for $s>a$.
$\mathcal{L}[e^{at}\sin(bt)](s)=\dfrac{b}{(s-a)^2+b^2}$	for $s>a$.
$\mathcal{L}[t^n j(t)](s)=(-1)^n J^{(n)}(s)$	where $J(s)=\mathcal{L}[j(t)](s)$.
$\mathcal{L}[e^{at}j(t)](s)=J(s-a)$	where $J(s)=\mathcal{L}[j(t)](s)$.
$\mathcal{L}[u(t-c)j(t-c)](s)=e^{-cs}J(s)$	where $J(s)=\mathcal{L}[j(t)](s)$ and u is the unit step function.
$\mathcal{L}[\delta(t-c)h(t)](s)=e^{-cs}h(c)$	where δ is the unit impluse.