Third In-Class Exam
Math 246, Professor David Levermore
Thursday, 15 November 2018

Your Name:____

UMD SID:____

Discussion Instructor (circle one): Sid Sharma Anqi Ye
Discussion Time (circle one): 8:00 9:00 10:00

No books, notes, calculators, or any electronic devices. If you need more space to answer a problem then use the back of one of these pages. **Do not separate the pages!** Indicate where the answer to each part of each problem is located. Cross out work that you do not want considered. **Your reasoning must be given for full credit!** Good luck!

Please copy and sign the University Honor Pledge below.
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University Honor Pledge: *I pledge on my honor that I have not given or received any unauthorized assistance on this examination.* ____

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Signature:____

Problem 1:	________/6	Problem 2:	________/10
Problem 3:	________/6	Problem 4:	________/10
Problem 5:	________/8	Problem 6:	________/8
Problem 7:	________/8	Problem 8:	________/8
Problem 9:	________/10	Problem 10:	________/8
Problem 11:	________/10	Problem 12:	________/8

Total Score: ________/100 Grade: ________

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(1) [6] Recast the ordinary differential equation $v'''' - v^2 v''' + \cos(v'') + e^v v' - t^4 = 0$ as a first-order system of ordinary differential equations.

(2) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 4 + t^6 \\ -t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} 3t^4 \\ 1 \end{pmatrix}$.

(a) [2] Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{B}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{B}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system found in part (b).

(d) [3] Compute the Green matrix associated with the system found in part (b).

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(3) [6] Given that -3 is an eigenvalue of the matrix

$$\mathbf{C} = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & -2 \\ 1 & 0 & 1 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{C}$ associated with -3.

(4) [10] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ 5 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

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(5) [8] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 19 liters and the second contains 24 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 3 liters per hour. Well-stirred brine flows from the first tank into the second at 4 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 6 liter per hour, and from the second into a drain at 2 liters per hour. At $t = 0$ there are 21 grams of salt in the first tank and 14 grams in the second.

(a) [6] Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(b) [2] Give the interval of definition for the solution of this initial-value problem.

(6) [8] A 3×3 matrix $\mathbf{H}$ has the eigenpairs

$$\left(-3\,,\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\,,\qquad\left(-1\,,\begin{pmatrix}1\\0\\-1\end{pmatrix}\right)\,,\qquad\left(2\,,\begin{pmatrix}1\\-2\\1\end{pmatrix}\right)\,.$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{H}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{H}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{H}\mathbf{x}$.

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(7) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(8) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

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(9) [10] Find the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4 + 10\mathrm{D}^2 + 9$.

(10) [8] Compute the Laplace transform of $f(t) = u(t-3)\, e^{-2t}$ from its definition. (Here u is the unit step function.)

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(11) [10] Consider the following (old style) MATLAB commands.

```
>> syms t s X; f = ['t^2 + heaviside(t − 3)*(6 − t^2) − heaviside(t − 5)*6'];
>> diffeqn = sym('D(D(x))(t) + 2*D(x)(t) + 10*x(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(x(t),t,s),t,s)', 'x(0)', 'D(x)(0)'}, {X, 3, −7});
>> xtrans = simplify(solve(algeqn, X));
>> x = ilaplace(xtrans, s, t)
```

(a) [2] Give the initial-value problem for $x(t)$ that is being solved.
(b) [8] Find the Laplace transform $X(s)$ of the solution $x(t)$. (DO NOT take the inverse Laplace transform of $X(s)$ to find $x(t)$, just solve for $X(s)$!)

You may refer to the table on the last page.

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[Y(s)](t)$ of the function

$$Y(s) = e^{-3s}\,\frac{s-6}{s^2+4s+20}\,.$$

You may refer to the table on the last page.

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Workspace (Give the number of the problem being worked!)

Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)$$

and u is the unit step function.

$$\mathcal{L}[\delta(t-c)h(t)](s) = e^{-cs}h(c) \qquad \text{where } \delta \text{ is the unit impluse}\,.$$