First In-Class Exam Solutions
Math 246, Professor David Levermore
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(1) [6] In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population such that it would triple every six weeks. There are 180,000 mosquitoes in the area when a flock of birds arrives that eats 40,000 mosquitoes per week.

(a) [4] Give an initial-value problem that governs $M(t)$, the number of mosquitoes in the area after the flock of birds arrives. (Do not solve the initial-value problem!)

(b) [2] Is the flock large enough to control the mosquitoes? (Why or why not?)

Solution (a). The population tripling every six weeks means that the growth rate r satisfies $e^{r6} = 3$, whereby $r = \frac{1}{6}\log(3)$. Therefore the initial-value problem that M satisfies is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \tfrac{1}{6}\log(3)M - 40,000\,, \qquad M(0) = 180,000\,.$$

Solution (b). Because the differential equation is autonomous (as well as linear), the monotonicity of $M(t)$ can be determined by a sign analysis of its right-hand side. We see from part (a) that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \tfrac{1}{6}\log(3)\left(M - \frac{240,000}{\log(3)}\right) \quad \text{is} \quad \begin{cases} < 0 & \text{for } M < \dfrac{240,000}{\log(3)}\,, \\ > 0 & \text{for } M > \dfrac{240,000}{\log(3)}\,. \end{cases}$$

This can be visualized with the phase-line portrait, which is

$$\begin{array}{ccc} - & & + \\ \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \; M \\ & \frac{240,000}{\log(3)} & \end{array}$$

Because $\log(x)$ is an increasing function and $3 > e$, we know that $\log(3) > \log(e) = 1$. Because 3 is close to e we know that $\log(3)$ is close to $\log(e) = 1$. Certainly we have $\log(3) < \frac{4}{3}$, which is equivalent to

$$M(0) = 180,000 < \frac{240,000}{\log(3)}\,.$$

This implies that $M(t)$ is a decreasing function of t. Therefore the flock is *large enough* to control the mosquitoes.

Remark. Because 3 is about 10% larger than e, we might expect that $\log(3)$ is also about 10% larger than $\log(e) = 1$. (In fact, $\frac{3}{e} \approx 1.10363832$ and $\log(3) \approx 1.09861229$.) This can be made precise as follows. The *concavity* of the function $f(x) = \log(x)$ over $x > 0$ implies that the graph of $f(x)$ lies below any tangent line. In particular, the graph of $f(x)$ lies below the tangent line approximation to $f(x)$ at the point $x = e$. This says that for any $x > 0$ we have

$$f(x) \leq f(e) + f'(e)(x - e)\,.$$

We expect that this tangent line approximation is pretty good when x is close to e. By setting $f(x) = \log(x)$ and $x = 3$, which is close to e, we obtain

$$\log(3) \leq \log(e) + \frac{1}{e}(3 - e) = 1 + \frac{3}{e} - 1 = \frac{3}{e}\,.$$

Because $\log(3) \approx 1.09861229$ and $\frac{3}{e} \approx 1.10363832$, this is a pretty good upper bound! In particular, because $e > \frac{27}{10}$ we see that

$$\log(3) \leq \frac{3}{e} < \frac{10}{9} < \frac{4}{3}\,.$$

You were not expected to make such an argument on the exam. Any argument that $\log(3)$ is closer to 1 than $\frac{4}{3}$ because 3 is close to e was fine.

(2) [22] Find an explicit solution for each of the following initial-value problems and give its interval of definition.

(a) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = (x - x^2)3t^2\,, \qquad x(0) = 2.$

Solution (a). This is a *nonautonomous, separable* equation. Its right-hand side is defined everywhere. Because $x - x^2 = x(1 - x)$, its only stationary points are $x = 0$ and $x = 1$. Because $x - x^2$ is differentiable at these stationary points, no other solution can touch them. Because its initial value 2 lies to the right of the stationary point 1, the solution $x(t)$ of the initial-value problem will lie to the right of the stationary point 1 for so long as it exists. In other words, the solution $x(t)$ must satisfy $x(t) > 1$ for every t in its interval of definition. To determine this interval of definition we must find $x(t)$.

The differential equation has the separated differential form

$$\frac{1}{x - x^2}\,\mathrm{d}x = 3t^2\,\mathrm{d}t\,,$$

whereby

$$\int \frac{1}{x - x^2}\,\mathrm{d}x = \int 3t^2\,\mathrm{d}t = t^3 + c_1\,.$$

By the residual (cover up) method we have the partial fraction identity

$$\frac{1}{x - x^2} = \frac{1}{x(1 - x)} = \frac{1}{x} + \frac{1}{1 - x}\,.$$

This identity plus the fact that $|x| = x$ and $|1 - x| = x - 1$ when $x > 1$ yield

$$\begin{aligned}
\int \frac{1}{x - x^2}\,\mathrm{d}x &= \int \frac{1}{x}\,\mathrm{d}x + \int \frac{1}{1 - x}\,\mathrm{d}x \\
&= \log(|x|) - \log(|1 - x|) + c_2 \\
&= \log(x) - \log(x - 1) + c_2 = \log\left(\frac{x}{x - 1}\right) + c_2\,.
\end{aligned}$$

By setting $c = c_1 - c_2$ we obtain the implicit general solution

$$\log\left(\frac{x}{x - 1}\right) = t^3 + c\,.$$

The initial condition $x(0) = 2$ implies that

$$\log\left(\frac{2}{2-1}\right) = 0^3 + c\,,$$

whereby $c = \log(2)$. Hence, the solution is governed implicitly by

$$\log\left(\frac{x}{x-1}\right) = t^3 + \log(2)\,.$$

Upon exponentiating both sides we obtain

$$\frac{x}{x-1} = e^{\log(2)+t^3} = 2e^{t^3}\,,$$

which becomes the linear expression in x given by

$$x = 2e^{t^3}(x-1)\,.$$

This can be solved to arrive at the explicit solution

$$x = \frac{2e^{t^3}}{2e^{t^3}-1}\,.$$

Because the denominator is positive at the initial time $t = 0$, the interval of definition is determined by $2e^{t^3} - 1 > 0$. Because $e^{t^3} > \frac{1}{2}$ implies $t^3 > -\log(2)$, we see that the *interval of definition* is

$$t > -\big(\log(2)\big)^{\frac{1}{3}}\,,$$

or equivalently, in interval notaion

$$\big(-\big(\log(2)\big)^{\frac{1}{3}}\,,\,\infty\big)\,.$$

(b) $(1+z^2)\dfrac{\mathrm{d}y}{\mathrm{d}z} + 6zy = \dfrac{3}{1+z^2}\,, \qquad y(0) = 2.$

Solution (b). This is a *nonhomogeneous linear* equation. Its normal form is

$$\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{6z}{1+z^2}\,y = \frac{3}{(1+z^2)^2}\,.$$

Its coefficient $6z/(1+z^2)$ and forcing $3/(1+z^2)^2$ both are continuous everywhere. Therefore the *interval of definition* of the solution is $(-\infty, \infty)$.

An integrating factor is

$$\exp\left(\int_0^z \frac{6s}{1+s^2}\,\mathrm{d}s\right) = \exp\big(3\log(1+z^2)\big) = (1+z^2)^3\,,$$

whereby the equation has the integrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}z}\big((1+z^2)^3 y\big) = (1+z^2)^3\frac{3}{(1+z^2)^2} = 3 + 3z^2\,.$$

By integrating both sides of this equation we find that

$$(1+z^2)^3 y = \int 3 + 3z^2\,\mathrm{d}z = 3z + z^3 + c\,.$$

The initial condition $y(0) = 2$ implies that $(1 + 0^2)^3 2 = 3 \cdot 0 + 0^3 + c$, whereby $c = 2$. Therefore the solution is

$$y = \frac{3z + z^3 + 2}{(1 + z^2)^3} \,.$$

This foumula confirms that its *interval of definition* is $(-\infty, \infty)$.

(3) [12] Consider the differential equation $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{(v^2 - 1)(v - 3)^2}{(v^2 + 1)(v + 3)^2}$.

(a) [7] Sketch its phase-line portrait over the interval $-4 \leq v \leq 4$. Identify with $\circ$ points where it has no solution. Identify with $\bullet$ its stationary points and classify each as being either stable, unstable, or semistable.

(b) [5] For each stationary point identify the set of initial-values $v(0)$ such that the solution $v(t)$ converges to that stationary point as $t \to \infty$.

Solution (a). This equation is autonomous. Its right-hand side is undefined at $v = -3$ and is differentiable elsewhere. Its stationary points are found by setting

$$\frac{(v^2 - 1)(v - 3)^2}{(v^2 + 1)(v + 3)^2} = 0 \,.$$

Because $v^2 - 1 = (v + 1)(v - 1)$, the stationary points are $v = -1$, $v = 1$, and $v = 3$. (Notice that $v^2 + 1 > 0$.) Because the right-hand side is differentiable at each of these stationary points, no other solutions will touch them. (Uniqueness!)

A sign analysis of the right-hand side shows that the phase-line portrait is

$$\begin{array}{ccccccccccccc}
 & & + & & + & & - & & + & & + & & \\
\to\to & + & \to\to & \circ & \to\to\to\to & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \to\to\to\to & \bullet & \to\to & + & \to\to \quad v \\
 & -4 & & -3 & & -1 & & 1 & & 3 & & 4 & \\
 & & & \text{undefined} & & \text{stable} & & \text{unstable} & & \text{semistable} & & &
\end{array}$$

Remark. The terms stable, unstable, and semistable are applied only to solutions. The point $v = -3$ is not a solution, so these terms should not be applied to it.

Solution (b). As t increases the solutions $v(t)$ will move in *the direction of the arrows* that are shown in the phase-line portrait given in the solution to part (a). Moreover, uniqueness implies that a nonstationary solution will not touch any stationary one.

- The phase-line portrait shows that for the stable stationary point -1 we have $v(t) \to -1$ as $t \to \infty$ if and only if $v(0)$ is in the interval $(-3, 1)$.
- The phase-line portrait shows that for the unstable stationary point 1 we have $v(t) \to 1$ as $t \to \infty$ if and only if $v(0) = 1$.
- The phase-line portrait shows that for the semistable stationary point 3 we have $v(t) \to 3$ as $t \to \infty$ if and only if $v(0)$ is in the interval $(1, 3]$.

(4) [12] Consider the following MATLAB function M-file.

```
function [t,u] = solveit(tI, uI, tF, n)

t = zeros(n + 1, 1); u = zeros(n + 1, 1);
t(1) = tI; u(1) = uI; h = (tF - tI)/n; hhalf = h/2;
for k = 1:n
t(k + 1) = t(k) + h;
fnow = (u(k))^3 + exp(t(k)*u(k)); uplus = u(k) + h*fnow;
fplus = (uplus)^3 + exp(t(k+1)*uplus); u(k + 1) = u(k) + hhalf*(fnow + fplus);
end
```

Suppose the input values are tI = 2, uI = 0, tF = 6, and n = 40.
(a) [4] What initial-value problem is being approximated numerically?
(b) [2] What is the numerical method being used?
(c) [2] What is the step size?
(d) [4] What will be the output values of t(2) and u(2)?

Solution (a). The initial-value problem being approximated numerically is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u^3 + \exp(tu) \,, \qquad u(2) = 0 \,.$$

Remark. An initial-value problem consists of both a differential equation and an initial condition. Both must be given for full credit.

Solution (b). The solution is being approximated by the Runge-trapezoidal method. (This is clear from the "hhalf*(fnow + fplus)" in last line of the "for" loop.)

Solution (c). Because tF = 6, tI = 2, and n = 40, the step size is

$$\mathrm{h} = \frac{\mathrm{tF} - \mathrm{tI}}{\mathrm{n}} = \frac{6-2}{40} = \frac{4}{40} = \frac{1}{10} = 0.1 \,.$$

Remark. The correct values for tF, tI, and n had to be plugged in to get full credit.

Solution (d). Because h = 0.1, we have hhalf = 0.05.
Because tI = 2 and uI = 0, we have t(1) = tI = 2, and x(1) = xI = 0.
Setting k = 1 inside the "for" loop then yields

$$\mathrm{t}(2) = \mathrm{t}(1) + \mathrm{h} = 2 + 0.1 = 2.1 \,,$$

$$\mathrm{fnow} = (\mathrm{u}(1))^3 + \exp(\mathrm{t}(1) * \mathrm{u}(1)) = 0^3 + \exp(2 \cdot 0) = 0 + 1 = 1 \,,$$

$$\mathrm{uplus} = \mathrm{u}(1) + \mathrm{h} * \mathrm{fnow} = 0 + 0.1 \cdot 1 = 0.1 \,,$$

$$\mathrm{fplus} = \mathrm{uplus}^3 + \exp(\mathrm{t}(2) * \mathrm{uplus}) = (0.1)^3 + \exp(2.1 \cdot 0.1) = (0.1)^3 + \exp(0.21) \,,$$

$$\mathrm{u}(2) = \mathrm{u}(1) + \mathrm{hhalf}*(\mathrm{fnow} + \mathrm{fplus}) = 0 + 0.05 \left(1 + (0.1)^3 + \exp(0.21)\right) .$$

Remark. This expression for u(2) did not have to be simplified to get full credit.

(5) [6] Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}k}{\mathrm{d}t} + \frac{\cos(t)}{\sin(t)}\, k = \frac{t}{t^2 - 16}\,, \qquad k(-5) = 4\,.$$

(Do not solve the equation to answer this question, but give reasoning!)

Solution. This problem is linear in k. It is already in normal form. The interval of definition can be read off as follows.

- First, notice that the coefficient $\cos(t)/\sin(t)$ is undefined at $t = n\pi$ for every integer n and is continuous elsewhere.
- Next, notice that the forcing $t/(t^2 - 16)$ is undefined at $t = \pm 4$ and is continuous elsewhere.

Therefore the *interval of definition* is $(-2\pi, -4)$ because

- the initial time $t = -5$ is in $(-2\pi, -4)$,
- both the coefficient and forcing are continuous over $(-2\pi, -4)$,
- the coefficient is undefined at $t = -2\pi$,
- the forcing is undefined at $t = -4$.

(6) [6] Sketch the graph that would be produced by the following Matlab commands.

```
[X, Y] = meshgrid(−5:0.1:5,−5:0.1:5)
contour(X, Y, X − Y.ˆ2, [−4, 0, 4])
axis square
```

Solution. The meshgrid command says the sketch should show both x and y axes marked from -5 to 5. The contour command plots the graph of the curve $x - y^2 = c$ for the values $c = -4$, $c = 0$, and $c = 4$, which are the graphs of the three parabolas

$$x = y^2 - 4\,, \qquad x = y^2\,, \qquad x = y^2 + 4\,.$$

A sketch will be shown during discussion.

(7) [8] Suppose you have used a numerical method to approximate the solution of an initial-value problem over the time interval $[1, 10]$ with 900 uniform time steps. What step size is needed to reduce the global error of your approximation by a factor of $\frac{1}{625}$ if the method you had used was each of the following? (Notice that $625 = 5^4$.)

(a) Runge-Kutta method

(b) Runge-midpoint method

(c) Runge-trapezoidal method

(d) explicit Euler method

Remark. Notice that the step size used in the original calculation is

$$h = \frac{t_F - t_I}{N} = \frac{10-1}{900} = \frac{1}{100}\,.$$

Solution (a). The Runge-Kutta method is fourth order, so its error scales like h^4. To reduce the error by a factor of $\frac{1}{625}$, we must reduce h by a factor of $\frac{1}{625}^{\frac{1}{4}} = \frac{1}{5}$. Because the original h was $\frac{1}{100}$, we must set

$$h = \frac{1}{100} \cdot \frac{1}{5} = \frac{1}{500}\,.$$

Solution (b). The Runge-midpoint method is second order, so its error scales like h^2. To reduce the error by a factor of $\frac{1}{625}$, we must reduce h by a factor of $\frac{1}{625}^{\frac{1}{2}} = \frac{1}{25}$. Because the original h was $\frac{1}{100}$, we must set

$$h = \frac{1}{100} \cdot \frac{1}{25} = \frac{1}{2500}\,.$$

Solution (c). The Runge-trapezoidal method is second order, so its error scales like h^2. To reduce the error by a factor of $\frac{1}{625}$, we must reduce h by a factor of $\frac{1}{625}^{\frac{1}{2}} = \frac{1}{25}$. Because the original h was $\frac{1}{100}$, we must set

$$h = \frac{1}{100} \cdot \frac{1}{25} = \frac{1}{2500}\,.$$

Solution (d). The explicit Euler method is first order, so its error scales like h. To reduce the error by a factor of $\frac{1}{625}$, we must reduce h by a factor of $\frac{1}{625}$. Because the original h was $\frac{1}{100}$, we must set

$$h = \frac{1}{100} \cdot \frac{1}{625} = \frac{1}{62500}\,.$$

Remark. The number of time steps needed to reduce the error by a factor of $\frac{1}{625}$ is respectively

$$\text{(a) } 900 \cdot 5\,, \qquad \text{(b) } 900 \cdot 25\,, \qquad \text{(c) } 900 \cdot 25\,, \qquad \text{(d) } 900 \cdot 625\,.$$

Had you computed these then the associated step sizes can be expressed as

$$\text{(a) } \frac{10-1}{900 \cdot 5}\,, \qquad \text{(b) } \frac{10-1}{900 \cdot 25}\,, \qquad \text{(c) } \frac{10-1}{900 \cdot 25}\,, \qquad \text{(d) } \frac{10-1}{900 \cdot 625}\,.$$

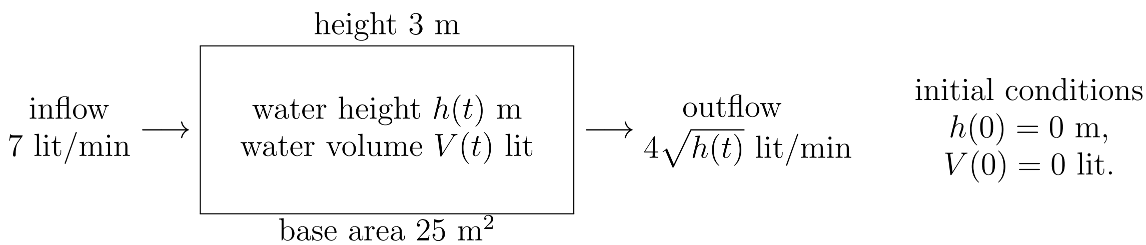
The arithmetic did not have to be carried out for full credit.

(8) [8] A tank has a square base with 5 meter edges, a height of 3 meters, and an open top. It is initially empty when water begins to fill it at a rate of 7 liters per minute. The water also drains from the tank through a hole in its bottom at a rate of $4\sqrt{h}$ liters per minute where $h(t)$ is the height of the water in the tank in meters.

(a) [6] Give an initial-value problem that governs $h(t)$. (Recall 1 m^3 = 1000 lit.) (Do not solve the initial-value problem!)

(b) [2] Does the tank overflow? (Why or why not?)

Solution (a). Let $V(t)$ be the volume (lit) of water in the tank at time t minutes. We have the following (optional) picture.



We want to write down an initial-value problem that governs $h(t)$.

Because the tank has a base with an area of 25 m^2, the volume of water in the tank is $25h(t)$ m^3. Because 1 m^3 = 1000 lit, $V(t) = 1000 \cdot 25h(t) = 25000h(t)$ lit. Because

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{RATE IN} - \text{RATE OUT} = 7 - 4\sqrt{h}\,,$$

and $V = 25000h$, the initial-value problem that governs $h(t)$ is

$$25000\,\frac{\mathrm{d}h}{\mathrm{d}t} = 7 - 4\sqrt{h}\,, \qquad h(0) = 0\,.$$

Each term in the differential equation has units of lit/min.

Solution (b). The differential equation is autonomous. Its right-hand side is defined for $h \geq 0$ and is differentiable for $h > 0$. It has one stationary point at $h = \frac{49}{16}$. Its phase-line portrait for $h > 0$ is

$$\begin{array}{ccc} + & & - \\ \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\; h \\ & \frac{49}{16} & \end{array}$$

This portrait shows that if $h(0) = 0$ then $h(t) \to \frac{49}{16}$ as $t \to \infty$. Because the height of the tank is 3 and $\frac{49}{16} > 3$, the tank will overflow.

(9) [20] For each of the following differential forms determine if it is exact or not. If it is exact then give an implicit general solution. Otherwise find an integrating factor. (You do not need to find a general solution in the last case.)

(a) $\big(\cos(x+y) - e^x\big)\,\mathrm{d}x + \big(\cos(x+y) + 2 + y^2\big)\,\mathrm{d}y = 0\,.$

Solution (b). This differential form is *exact* because

$$\partial_y\big(\cos(x+y) - e^x\big) = -\sin(x+y) \quad = \quad \partial_x\big(\cos(x+y) + 2 + y^2\big) = -\sin(x+y)\,.$$

Therefore we can find $H(x,y)$ such that

$$\partial_x H(x,y) = \cos(x+y) - e^x\,, \qquad \partial_y H(x,y) = \cos(x+y) + 2 + y^2\,.$$

Integrating the first equation with respect to x shows

$$H(x,y) = \sin(x+y) - e^x + h(y)\,,$$

which implies that

$$\partial_y H(x,y) = \cos(x+y) + h'(y)\,.$$

Plugging this expression for $\partial_y H(x,y)$ into the second equation gives

$$\cos(x+y) + h'(y) = \cos(x+y) + 2 + y^2\,,$$

which shows $h'(y) = 2 + y^2$. Taking $h(y) = 2y + \frac{1}{3}y^3$, an implicit general solution is

$$\sin(x+y) - e^x + 2y + \tfrac{1}{3}y^3 = c\,.$$

(b) $(y^3 + 4x^3y)\,\mathrm{d}x + (5xy^2 + 3x^4)\,\mathrm{d}y = 0\,.$

Solution (b). This differential form is *not exact* because

$$\partial_y(y^3 + 4x^3y) = 3y^2 + 4x^3 \quad \neq \quad \partial_x(5xy^2 + 3x^4) = 5y^2 + 12x^3\,.$$

We seek an integrating factor ρ that satisfies

$$\partial_y\big[(y^3 + 4x^3y)\rho\big] = \partial_x\big[(5xy^2 + 3x^4)\rho\big]\,.$$

Expanding the partial derivatives yields

$$(y^3 + 4x^3y)\partial_y\rho + (3y^2 + 4x^3)\rho = (5xy^2 + 3x^4)\partial_x\rho + (5y^2 + 12x^3)\rho\,.$$

Grouping the ρ terms together gives

$$(y^3 + 4x^3y)\partial_y\rho = (5xy^2 + 3x^4)\partial_x\rho + (2y^2 + 8x^3)\rho\,.$$

If we set $\partial_x\rho = 0$ then this reduces to $y\partial_y\rho = 2\rho$, which yields the integrating factor $\rho = y^2$.

Remark. Because the differential form was not exact, all we were asked to do was find an integrating factor. If we had been asked to find an implicit general solution then we would seek $H(x,y)$ such that

$$\partial_x H(x,y) = y^5 + 4x^3y^3\,, \qquad \partial_y H(x,y) = 5xy^4 + 3x^4y^2\,.$$

These equations can be integrated to find $H(x,y) = xy^5 + x^4y^3$. Therefore an implicit general solution is

$$xy^5 + x^4y^3 = c\,.$$