

**Solutions of the Sample Problems for the Third In-Class Exam  
Math 246, Spring 2018, Professor David Levermore**

(1) Compute the Laplace transform of  $f(t) = te^{3t}u(t-2)$  from its definition.

**Solution.** The definition of the Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T \rightarrow \infty} \int_0^T e^{-st} te^{3t}u(t-2) dt = \lim_{T \rightarrow \infty} \int_2^T te^{-(s-3)t} dt.$$

This limit diverges to  $+\infty$  for  $s \leq 3$  because in that case for every  $T > 2$  we have

$$\int_2^T te^{-(s-3)t} dt \geq \int_2^T t dt = \frac{T^2}{2} - 2,$$

which clearly diverges to  $+\infty$  as  $T \rightarrow \infty$ .

For  $s > 3$  an integration by parts shows that

$$\begin{aligned} \int_2^T te^{-(s-3)t} dt &= -t \frac{e^{-(s-3)t}}{s-3} \Big|_2^T + \int_2^T \frac{e^{-(s-3)t}}{s-3} dt \\ &= \left( -t \frac{e^{-(s-3)t}}{s-3} - \frac{e^{-(s-3)t}}{(s-3)^2} \right) \Big|_2^T \\ &= \left( -T \frac{e^{-(s-3)T}}{s-3} - \frac{e^{-(s-3)T}}{(s-3)^2} \right) + \left( 2 \frac{e^{-(s-3)2}}{s-3} + \frac{e^{-(s-3)2}}{(s-3)^2} \right). \end{aligned}$$

Hence, for  $s > 3$  we have that

$$\begin{aligned} \mathcal{L}[f](s) &= \lim_{T \rightarrow \infty} \left[ \left( -T \frac{e^{-(s-3)T}}{s-3} - \frac{e^{-(s-3)T}}{(s-3)^2} \right) + \left( 2 \frac{e^{-(s-3)2}}{s-3} + \frac{e^{-(s-3)2}}{(s-3)^2} \right) \right] \\ &= \frac{e^{-(s-3)2}}{(s-3)^2} + 2 \frac{e^{-(s-3)2}}{s-3} - \lim_{T \rightarrow \infty} \left( T \frac{e^{-(s-3)T}}{s-3} + \frac{e^{-(s-3)T}}{(s-3)^2} \right) \\ &= \frac{e^{-(s-3)2}}{(s-3)^2} + 2 \frac{e^{-(s-3)2}}{s-3}. \end{aligned}$$

(2) Consider the following (old style) MATLAB commands.

```
>> syms t s Y; f = ['heaviside(t)*t^2 + heaviside(t - 3)*(3*t - t^2)'];
>> diffeqn = sym('D(D(y))(t) - 6*D(y)(t) + 10*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s'}, 'y(0)', 'D(y)(0)');
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

- (a) Give the initial-value problem for  $y(t)$  that is being solved.  
 (b) Find the Laplace transform  $Y(s)$  of the solution  $y(t)$ .

DO NOT take the inverse Laplace transform of  $Y(s)$  to find  $y(t)$ , just solve for  $Y(s)$ ! You may refer to the table on the last page.

**Solution (a).** The initial-value problem for  $y(t)$  that is being solved is

$$y'' - 6y' + 10y = f(t), \quad y(0) = 2, \quad y'(0) = 3,$$

where the forcing  $f(t)$  can be expressed either as

$$f(t) = \begin{cases} t^2 & \text{for } 0 \leq t < 3, \\ 3t & \text{for } 3 \leq t, \end{cases}$$

or in terms of the unit step function as  $f(t) = t^2 + u(t-3)(3t - t^2)$ .

**Solution (b).** The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) - 6\mathcal{L}[y'](s) + 10\mathcal{L}[y](s) = \mathcal{L}[f](s),$$

where

$$\mathcal{L}[y](s) = Y(s),$$

$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s) - y(0) = sY(s) - 2,$$

$$\mathcal{L}[y''](s) = s\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - 2s - 3.$$

To compute  $\mathcal{L}[f](s)$ , we first write  $f(t)$  as

$$f(t) = t^2 + u(t-3)(3t - t^2) = t^2 + u(t-3)j(t-3),$$

where by setting  $j(t-3) = 3t - t^2$  we see by the shifty step method that

$$j(t) = 3(t+3) - (t+3)^2 = 3t + 9 - t^2 - 6t - 9 = -t^2 - 3t.$$

Referring to the table on the last page, item 1 with  $a = 0$  and  $n = 2$  and with  $a = 0$  and  $n = 1$  shows that

$$\mathcal{L}[t^2](s) = \frac{2}{s^3}, \quad \mathcal{L}[t](s) = \frac{1}{s^2},$$

whereby item 6 with  $c = 3$  and  $j(t) = -t^2 - 3t$  shows that

$$\mathcal{L}[u(t-3)j(t-3)](s) = e^{-3s}\mathcal{L}[j](s) = -e^{-3s}\mathcal{L}[t^2 + 3t](s) = -e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore

$$\mathcal{L}[f](s) = \mathcal{L}[t^2 + u(t-3)j(t-3)](s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

The Laplace transform of the initial-value problem then becomes

$$(s^2Y(s) - 2s - 3) - 6(sY(s) - 2) + 10Y(s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right),$$

which becomes

$$(s^2 - 6s + 10)Y(s) - 2s + 9 = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore  $Y(s)$  is given by

$$Y(s) = \frac{1}{s^2 - 6s + 10}\left(2s - 9 + \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right)\right).$$

(3) Find  $Y(s) = \mathcal{L}[y](s)$  where  $y(t)$  solves the initial-value problem

$$y'' + 4y' + 13y = f(t), \quad y(0) = 4, \quad y'(0) = 1,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \leq t < 2\pi, \\ t - 2\pi & \text{for } t \geq 2\pi. \end{cases}$$

DO NOT take the inverse Laplace transform of  $Y(s)$  to find  $y(t)$ , just solve for  $Y(s)$ ! You may refer to the table on the last page.

**Solution.** The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 13\mathcal{L}[y](s) = \mathcal{L}[f](s),$$

where

$$\mathcal{L}[y](s) = Y(s),$$

$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s) - y(0) = sY(s) - 4,$$

$$\mathcal{L}[y''](s) = s\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - 4s - 1.$$

To compute  $\mathcal{L}[f](s)$ , first write  $f$  as

$$\begin{aligned} f(t) &= (1 - u(t - 2\pi)) \cos(t) + u(t - 2\pi)(t - 2\pi) \\ &= \cos(t) + u(t - 2\pi)(t - 2\pi - \cos(t)) \\ &= \cos(t) + u(t - 2\pi)j(t - 2\pi), \end{aligned}$$

where by setting  $j(t - 2\pi) = t - 2\pi - \cos(t)$  we see by the shifty step method that

$$j(t) = (t + 2\pi) - 2\pi - \cos(t + 2\pi) = t - \cos(t).$$

Here we have used the fact that  $\cos(t)$  is  $2\pi$ -periodic. Referring to the table on the last page, item 6 with  $c = 2\pi$  shows that

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}[\cos(t)](s) + \mathcal{L}[u(t - 2\pi)j(t - 2\pi)](s) \\ &= \mathcal{L}[\cos(t)](s) + e^{-2\pi s} \mathcal{L}[j(t)](s) \\ &= \mathcal{L}[\cos(t)](s) + e^{-2\pi s} \mathcal{L}[t - \cos(t)](s). \end{aligned}$$

Then item 2 with  $a = 0$  and  $b = 1$ , and item 1 with  $n = 1$  and  $a = 1$  imply that

$$\mathcal{L}[f](s) = \frac{s}{s^2 + 1} + e^{-2\pi s} \left( \frac{1}{s^2} - \frac{s}{s^2 + 1} \right).$$

The Laplace transform of the initial-value problem then becomes

$$(s^2Y(s) - 4s - 1) + 4(sY(s) - 4) + 13Y(s) = (1 - e^{-2\pi s}) \frac{s}{s^2 + 1} + e^{-2\pi s} \frac{1}{s^2},$$

which becomes

$$(s^2 + 4s + 13)Y(s) - 4s - 1 - 16 = (1 - e^{-2\pi s}) \frac{s}{s^2 + 1} + e^{-2\pi s} \frac{1}{s^2}.$$

Hence,  $Y(s)$  is given by

$$Y(s) = \frac{1}{s^2 + 4s + 13} \left( 4s + 17 + (1 - e^{-2\pi s}) \frac{s}{s^2 + 1} + e^{-2\pi s} \frac{1}{s^2} \right).$$

(4) Find  $X(s) = \mathcal{L}[x](s)$  where  $x(t)$  solves the initial-value problem

$$x'' + 4x = \delta(t - 3), \quad x(0) = 5, \quad x'(0) = 0.$$

DO NOT take the inverse Laplace transform of  $X(s)$  to find  $x(t)$ , just solve for  $X(s)$ ! You may refer to the table on the last page.

**Solution.** The Laplace transform of the initial-value problem is

$$\mathcal{L}[x''](s) + 4\mathcal{L}[x](s) = \mathcal{L}[\delta(t - 3)](s),$$

where

$$\mathcal{L}[x](s) = X(s),$$

$$\mathcal{L}[x'](s) = s\mathcal{L}[x](s) - x(0) = sX(s) - 5,$$

$$\mathcal{L}[x''](s) = s\mathcal{L}[x'](s) - x'(0) = s^2X(s) - 5s - 0.$$

Referring to the table on the last page, item 7 with  $c = 3$  and  $h(t) = 1$  shows that

$$\mathcal{L}[\delta(t - 3)](s) = e^{-3s}.$$

The Laplace transform of the initial-value problem then becomes

$$(s^2 + 4)X(s) - 5s = e^{-3s}.$$

Hence,  $X(s)$  is given by

$$X(s) = \frac{5s + e^{-3s}}{s^2 + 4}.$$

**Remark.** You should be able to take the inverse Laplace transform to obtain

$$x(t) = \mathcal{L}^{-1}[X](t) = \mathcal{L}^{-1}\left[\frac{5s + e^{-3s}}{s^2 + 4}\right](t) = 5\cos(2t) + \frac{1}{2}u(t - 3)\sin(2(t - 3)).$$

(5) Find the inverse Laplace transforms of the following functions.

(a)  $F(s) = \frac{2}{(s + 5)^2},$

(b)  $F(s) = \frac{3s}{s^2 - s - 6},$

(c)  $F(s) = \frac{(s - 2)e^{-3s}}{s^2 - 4s + 5}.$

You may refer to the table on the last page.

**Solution (a).** Referring to the table on the last page, item 1 with  $n = 1$  and  $a = -5$  gives

$$\mathcal{L}[te^{-5t}](s) = \frac{1}{(s + 5)^2}.$$

Therefore we conclude that

$$\mathcal{L}^{-1}\left[\frac{2}{(s + 5)^2}\right](t) = 2\mathcal{L}^{-1}\left[\frac{1}{(s + 5)^2}\right](t) = 2te^{-5t}.$$

**Solution (b).** Because the denominator factors as  $(s-3)(s+2)$ , we have the partial fraction identity

$$\frac{3s}{s^2 - s - 6} = \frac{3s}{(s-3)(s+2)} = \frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}.$$

Referring to the table on the last page, item 1 with  $n = 0$  and  $a = 3$ , and with  $n = 0$  and  $a = -2$  gives

$$\mathcal{L}[e^{3t}](s) = \frac{1}{s-3}, \quad \mathcal{L}[e^{-2t}](s) = \frac{1}{s+2}.$$

Therefore we conclude that

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{3s}{s^2 - s - 6}\right](t) &= \mathcal{L}^{-1}\left[\frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}\right](t) \\ &= \frac{9}{5}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right](t) + \frac{6}{5}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t) \\ &= \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}. \end{aligned}$$

**Solution (c).** Complete the square in the denominator to get  $(s-2)^2 + 1$ . Referring to the table on the last page, item 2 with  $a = 2$  and  $b = 1$  gives

$$\mathcal{L}[e^{2t} \cos(t)](s) = \frac{s-2}{(s-2)^2 + 1}.$$

Item 6 with  $c = 3$  and  $j(t) = e^{2t} \cos(t)$  then gives

$$\mathcal{L}[u(t-3)e^{2(t-3)} \cos(t-3)](s) = e^{-3s} \frac{s-2}{(s-2)^2 + 1}.$$

Therefore we conclude that

$$\mathcal{L}^{-1}\left[e^{-3s} \frac{s-2}{s^2 - 4s + 5}\right](t) = u(t-3)e^{2(t-3)} \cos(t-3).$$

(6) For each of the following differential operators compute its Green function  $g(t)$  and its natural fundamental set for  $t = 0$ .

(a)  $L = D^4 + 8D^2 - 9$ ,

(b)  $L = (D - 2)^3$ .

You may refer to the table on the last page.

**Solution (a).** The characteristic polynomial of  $L = D^4 + 8D^2 - 9$  is  $p(s) = s^4 + 8s^2 - 9$ . Therefore its Green function  $g(t)$  is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s^4 + 8s^2 - 9}\right](t).$$

Because  $p(s)$  factors as  $p(s) = (s^2 - 1)(s^2 + 9)$  we have the partial fraction identity

$$\frac{1}{s^4 + 8s^2 - 9} = \frac{1}{(s^2 - 1)(s^2 + 9)} = \frac{\frac{1}{10}}{s^2 - 1} + \frac{-\frac{1}{10}}{s^2 + 9}.$$

Because  $s^2 - 1$  factors as  $s^2 - 1 = (s - 1)(s + 1)$  we have the partial fraction identity

$$\frac{1}{s^2 - 1} = \frac{1}{(s - 1)(s + 1)} = \frac{\frac{1}{2}}{s - 1} + \frac{-\frac{1}{2}}{s + 1}.$$

By combining the above partial fraction identities we obtain

$$\frac{1}{s^4 + 8s^2 - 9} = \frac{1}{20} \frac{1}{s - 1} - \frac{1}{20} \frac{1}{s + 1} - \frac{1}{10} \frac{1}{s^2 + 9}.$$

Referring to the table on the last page, item 1 with  $a = 1$  and  $n = 0$  and with  $a = -1$  and  $n = 0$  gives

$$\mathcal{L}^{-1}\left[\frac{1}{s - 1}\right](t) = e^t, \quad \mathcal{L}^{-1}\left[\frac{1}{s + 1}\right](t) = e^{-t},$$

while item 3 with  $a = 0$  and  $b = 3$  gives

$$\mathcal{L}^{-1}\left[\frac{3}{s^2 + 9}\right](t) = \sin(3t).$$

Therefore the Green function  $g(t)$  is given by

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^4 + 8s^2 - 9}\right](t) \\ &= \frac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s - 1}\right](t) - \frac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s + 1}\right](t) - \frac{1}{30}\mathcal{L}^{-1}\left[\frac{3}{s^2 + 9}\right](t) \\ &= \frac{1}{20}e^t - \frac{1}{20}e^{-t} - \frac{1}{30}\sin(3t). \end{aligned}$$

Then because we see the characteristic polynomial as

$$p(s) = s^4 + 0s^3 + 8s^2 + 0s - 9,$$

the natural fundamental set for  $t = 0$  is found by

$$\begin{aligned} N_3(t) &= g(t) = \frac{1}{20}e^t - \frac{1}{20}e^{-t} - \frac{1}{30}\sin(3t), \\ N_2(t) &= N_3'(t) + 0g(t) = \frac{1}{20}e^t + \frac{1}{20}e^{-t} - \frac{1}{10}\cos(3t), \\ N_1(t) &= N_2'(t) + 8g(t) \\ &= \frac{1}{20}e^t - \frac{1}{20}e^{-t} + \frac{3}{10}\sin(3t) + \frac{8}{20}e^t - \frac{8}{20}e^{-t} - \frac{8}{30}\sin(3t), \\ &= \frac{9}{20}e^t - \frac{9}{20}e^{-t} + \frac{1}{30}\sin(3t), \\ N_0(t) &= N_1'(t) + 0g(t) = \frac{9}{20}e^t + \frac{9}{20}e^{-t} + \frac{1}{10}\cos(3t). \end{aligned}$$

**Remark.** The calculation of the natural fundamental set is a bit simpler if the Green function is expressed in terms of hyperbolic functions. It becomes

$$\begin{aligned} N_3(t) &= g(t) = \frac{1}{10}\sinh(t) - \frac{1}{30}\sin(3t), \\ N_2(t) &= N_3'(t) + 0g(t) = \frac{1}{10}\cosh(t) - \frac{1}{10}\cos(3t), \\ N_1(t) &= N_2'(t) + 8g(t) \\ &= \frac{1}{10}\sinh(t) + \frac{3}{10}\sin(3t) + \frac{8}{10}\sinh(t) - \frac{8}{30}\sin(3t), \\ &= \frac{9}{10}\sinh(t) + \frac{1}{30}\sin(3t), \\ N_0(t) &= N_1'(t) + 0g(t) = \frac{9}{10}\cosh(t) + \frac{1}{10}\cos(3t). \end{aligned}$$

**Solution (b).** The characteristic polynomial of  $L = (D - 2)^3$  is  $p(s) = (s - 2)^3$ . Therefore its Green function  $g(t)$  is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s - 2)^3}\right](t).$$

Referring to the table on the last page, item 1 with  $a = 2$  and  $n = 2$  gives

$$g(t) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s - 2)^3}\right](t) = \frac{1}{2}t^2e^{2t}.$$

Then because by the binomial expansion we see the characteristic polynomial as

$$\begin{aligned} p(s) &= (s - 2)^3 = s^3 + 3(-2)s^2 + 3(-2)^2s + (-2)^3 \\ &= s^3 - 6s^2 + 12s - 8, \end{aligned}$$

the natural fundamental set for  $t = 0$  is found by

$$N_2(t) = g(t) = \frac{1}{2}t^2e^{2t},$$

$$N_1(t) = N_2'(t) - 6g(t) = (te^{2t} + t^2e^{2t}) - \frac{6}{2}t^2e^{2t} = te^{2t} - 2t^2e^{2t},$$

$$N_0(t) = N_1'(t) + 12g(t) = (e^{2t} - 2te^{2t} - 4t^2e^{2t}) + \frac{12}{2}t^2e^{2t} = e^{2t} - 2te^{2t} + 2t^2e^{2t}.$$

- (7) Recast the equation  $u''' + t^2u' - 3u = \sinh(2t)$  as a first-order system of ordinary differential equations.

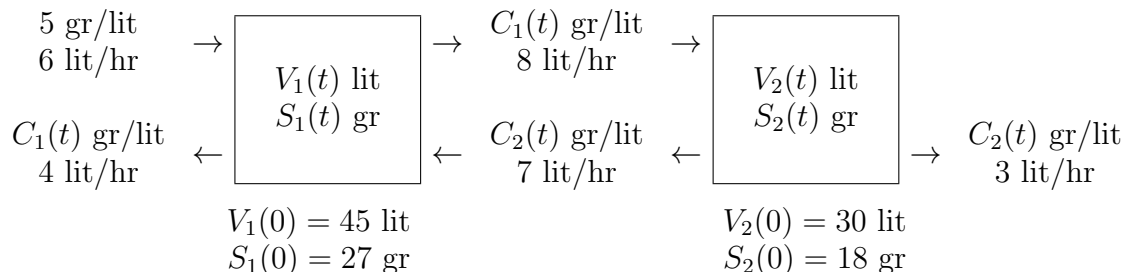
**Solution.** Because the equation is third order, the first-order system must have dimension three. The simplest such first-order system is

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \sinh(2t) + 3x_1 - t^2x_2 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}.$$

- (8) Two interconnected tanks are filled with brine (salt water). At  $t = 0$  the first tank contains 45 liters and the second contains 30 liters. Brine with a salt concentration of 5 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 8 liters per hour, from the second into the first at 7 liters per hour, from the first into a drain at 4 liter per hour, and from the second into a drain at 3 liters per hour. At  $t = 0$  there are 27 grams of salt in the first tank and 18 grams in the second.

- Give an initial-value problem that governs the amount of salt in each tank as a function of time.
- Give the interval of definition for the solution of this initial-value problem.

**Solution (a).** Let  $V_1(t)$  and  $V_2(t)$  be the volumes (lit) of brine in the first and second tank at time  $t$  hours. Let  $S_1(t)$  and  $S_2(t)$  be the mass (gr) of salt in the first and second tank at time  $t$  hours. Because mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time  $t$  are  $C_1(t) = S_1(t)/V_1(t)$  and  $C_2(t) = S_2(t)/V_2(t)$  respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.



We are asked to write down an initial-value problem that governs  $S_1(t)$  and  $S_2(t)$ .

The rates work out so there will be  $V_1(t) = 45 + t$  liters of brine in the first tank and  $V_2(t) = 30 - 2t$  liters in the second. Then  $S_1(t)$  and  $S_2(t)$  are governed by the initial-value problem

$$\begin{aligned}
 \frac{dS_1}{dt} &= 5 \cdot 6 + \frac{S_2}{30 - 2t} 7 - \frac{S_1}{45 + t} 8 - \frac{S_1}{45 + t} 4, & S_1(0) &= 27, \\
 \frac{dS_2}{dt} &= \frac{S_1}{45 + t} 8 - \frac{S_2}{30 - 2t} 7 - \frac{S_2}{30 - 2t} 3, & S_2(0) &= 18.
 \end{aligned}$$

You could leave the answer in the above form. However, it can be simplified to

$$\begin{aligned}
 \frac{dS_1}{dt} &= 30 + \frac{7}{30 - 2t} S_2 - \frac{12}{45 + t} S_1, & S_1(0) &= 27, \\
 \frac{dS_2}{dt} &= \frac{8}{45 + t} S_1 - \frac{5}{15 - t} S_2, & S_2(0) &= 18.
 \end{aligned}$$

**Solution (b).** This first-order system of differential equations is *linear*.

- ◇ Its coefficients are undefined either at  $t = -45$  or at  $t = 15$  and are continuous elsewhere.
- ◇ Its forcing is constant, so is continuous everywhere.
- ◇ Its initial time is  $t = 0$ .

Therefore the natural interval of definition for the solution of this initial-value problem is  $(-45, 15)$  because:

- the initial time  $t = 0$  is in  $(-45, 15)$ ;
- all the coefficients and the forcing are continuous over  $(-45, 15)$ ;
- two coefficients are undefined at  $t = -45$ ;
- two coefficients are undefined at  $t = 15$ .

However, it could also be argued that the interval of definition for the solution of this initial-value problem is  $[0, 15)$  because the word problem starts at  $t = 0$ .



(9) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

- (a)  $\mathbf{A}^T$ ,
- (b)  $\overline{\mathbf{A}}$ ,
- (c)  $\mathbf{A}^H$ ,
- (d)  $5\mathbf{A} - \mathbf{B}$ ,
- (e)  $\mathbf{AB}$ ,
- (f)  $\mathbf{B}^{-1}$ .

**Solution (a).** The transpose of  $\mathbf{A}$  is

$$\mathbf{A}^T = \begin{pmatrix} -i2 & 2+i \\ 1+i & -4 \end{pmatrix}.$$

**Solution (b).** The conjugate of  $\mathbf{A}$  is

$$\overline{\mathbf{A}} = \begin{pmatrix} i2 & 1-i \\ 2-i & -4 \end{pmatrix}.$$

**Solution (c).** The Hermitian transpose of  $\mathbf{A}$  is

$$\mathbf{A}^H = \begin{pmatrix} i2 & 2-i \\ 1-i & -4 \end{pmatrix}.$$

**Solution (d).** The difference of  $5\mathbf{A}$  and  $\mathbf{B}$  is given by

$$5\mathbf{A} - \mathbf{B} = \begin{pmatrix} -i10 & 5+i5 \\ 10+i5 & -20 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} -7-i10 & -1+i5 \\ 2+i5 & -27 \end{pmatrix}.$$

**Solution (e).** The product of  $\mathbf{A}$  and  $\mathbf{B}$  is given by

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -i2 \cdot 7 + (1+i) \cdot 8 & -i2 \cdot 6 + (1+i) \cdot 7 \\ (2+i) \cdot 7 - 4 \cdot 8 & (2+i) \cdot 6 - 4 \cdot 7 \end{pmatrix} \\ &= \begin{pmatrix} 8-i6 & 7-i5 \\ -18+i7 & -16+i6 \end{pmatrix}. \end{aligned}$$

**Solution (f).** Observe that it is clear that  $\mathbf{B}$  has an inverse because

$$\det(\mathbf{B}) = \det \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = 7 \cdot 7 - 6 \cdot 8 = 49 - 48 = 1.$$

Then the inverse of  $\mathbf{B}$  is given by

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix}.$$

(10) Consider the vector-valued functions  $\mathbf{x}_1(t) = \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix}$ ,  $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$ .

(a) Compute the Wronskian  $\text{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$ .

(b) Find  $\mathbf{A}(t)$  such that  $\mathbf{x}_1, \mathbf{x}_2$  is a fundamental set of solutions to  $\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}$  wherever  $\text{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$ .

(c) Give a fundamental matrix  $\Psi(t)$  for the system found in part (b).

(d) For the system found in part (b), solve the initial-value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(e) For the  $\mathbf{A}(t)$  found in part (b), give the Green matrix for the system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t).$$

**Solution (a).**

$$\text{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} = 3t^4 + 9 - 2t^4 = t^4 + 9.$$

**Solution (b).** Let  $\Psi(t) = \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix}$ . Because  $\frac{d\Psi(t)}{dt} = \mathbf{A}(t)\Psi(t)$ , we have

$$\begin{aligned} \mathbf{A}(t) &= \frac{d\Psi(t)}{dt} \Psi(t)^{-1} = \begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix} \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{t^4 + 9} \begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix} \begin{pmatrix} 3 & -t^2 \\ -2t^2 & t^4 + 3 \end{pmatrix} = \frac{1}{t^4 + 9} \begin{pmatrix} 8t^3 & 6t - 2t^5 \\ 12t & -4t^3 \end{pmatrix}. \end{aligned}$$

**Solution (c).** Because  $\mathbf{x}_1(t), \mathbf{x}_2(t)$  is a fundamental set of solutions to the system found in part (b), a fundamental matrix for the system found in part (b) is simply given by

$$\Psi(t) = (\mathbf{x}_1(t) \quad \mathbf{x}_2(t)) = \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix}.$$

**Solution (d).** Because a fundamental matrix  $\Psi(t)$  for the system found in part (b) was given in part (c), the solution of the initial-value problem is

$$\begin{aligned} \mathbf{x}(t) &= \Psi(t)\Psi(1)^{-1}\mathbf{x}(1) = \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \frac{1}{10} \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 3t^4 + 9 - 2t^2 \\ 6t^2 - 6 \end{pmatrix}. \end{aligned}$$

**Alternative Solution (d).** Because  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$  is a fundamental set of solutions to the system found in part (b), a general solution is given by

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) = c_1 \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 3 \end{pmatrix}.$$

The initial condition then implies that

$$\mathbf{x}(1) = c_1 \begin{pmatrix} 4 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4c_1 + c_2 \\ 2c_1 + 3c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

from which we see that  $c_1 = \frac{3}{10}$  and  $c_2 = -\frac{1}{5}$ . The solution of the initial-value problem is thereby

$$\mathbf{x}(t) = \frac{3}{10} \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} t^2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}t^4 - \frac{1}{5}t^2 + \frac{9}{10} \\ \frac{3}{5}t^2 - \frac{3}{5} \end{pmatrix}.$$

**Solution (e).** Because a fundamental matrix  $\Psi(t)$  for the system found in part (b) was given in part (c), the Green matrix for the nonhomogeneous system is

$$\begin{aligned} \mathbf{G}(t, s) &= \Psi(t)\Psi(s)^{-1} = \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} s^4 + 3 & s^2 \\ 2s^2 & 3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \frac{1}{s^4 + 9} \begin{pmatrix} 3 & -s^2 \\ -2s^2 & s^4 + 3 \end{pmatrix} \\ &= \frac{1}{s^4 + 9} \begin{pmatrix} 3t^4 + 9 - 2t^2s^2 & t^2(s^4 + 3) - (t^4 + 3)s^2 \\ 6t^2 - 6s^2 & 3s^4 + 9 - 2t^2s^2 \end{pmatrix}. \end{aligned}$$

(11) Compute  $e^{t\mathbf{A}}$  for the following matrices.

(a)  $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b)  $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

**Solution (a) by Two-by-Two Formula.** Because

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix},$$

the characteristic polynomial of  $\mathbf{A}$  is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z - 1)^2 - 2^2.$$

This is a difference of squares with  $\mu = 1$  and  $\nu = 2$ . Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[ \cosh(2t)\mathbf{I} + \frac{\sinh(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[ \cosh(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(2t)}{2} \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix}. \end{aligned}$$

**Solution (a) by the Natural Fundamental Set Method.** Because

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix},$$

the characteristic polynomial of  $\mathbf{A}$  is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z + 1)(z - 3).$$

Below we show that the natural fundamental set of solutions for  $t = 0$  associated with  $p(D) = D^2 - 2D + 3$  is

$$N_0(t) = \frac{e^{3t} + 3e^{-t}}{4}, \quad N_1(t) = \frac{e^{3t} - e^{-t}}{4}.$$

Then

$$\begin{aligned} e^{t\mathbf{A}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{A} = \frac{e^{3t} + 3e^{-t}}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{3t} - e^{-t}}{4} \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{3t} + 2e^{-t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}. \end{aligned}$$

**From the Green Function.** By the partial fraction identity

$$\frac{1}{s^2 - 2s + 3} = \frac{1}{(s - 3)(s + 1)} = \frac{\frac{1}{4}}{s - 3} + \frac{-\frac{1}{4}}{s + 1},$$

the Green function associated with  $p(D) = D^2 - 2D + 3$  is

$$\begin{aligned} g(t) &= \mathcal{L}^{-1} \left[ \frac{1}{p(s)} \right] (t) = \mathcal{L}^{-1} \left[ \frac{1}{s^2 - 2s + 3} \right] (t) \\ &= \frac{1}{4} \mathcal{L}^{-1} \left[ \frac{1}{s - 3} \right] (t) - \frac{1}{4} \mathcal{L}^{-1} \left[ \frac{1}{s + 1} \right] (t) = \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t}. \end{aligned}$$

Then, because the characteristic polynomial is  $p(s) = s^2 - 2s + 3$ , the natural fundamental set is

$$\begin{aligned} N_1(t) &= g(t) = \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t}, \\ N_0(t) &= N_1'(t) - 2g(t) = \left( \frac{3}{4} e^{3t} + \frac{1}{4} e^{-t} \right) - \left( \frac{2}{4} e^{3t} - \frac{2}{4} e^{-t} \right) = \frac{1}{4} e^{3t} + \frac{3}{4} e^{-t}. \end{aligned}$$

**From the General Initial-Value Problem.** The general initial-value problem associated with  $p(D) = D^2 - 2D + 3$  is

$$y'' - 2y' - 3y = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

This has the general solution  $y(t) = c_1 e^{3t} + c_2 e^{-t}$ . Because  $y'(t) = 3c_1 e^{3t} - c_2 e^{-t}$ , the general initial conditions yield

$$y_0 = y(0) = c_1 + c_2, \quad y_1 = y'(0) = 3c_1 - c_2.$$

This system can be solved to obtain

$$c_1 = \frac{y_0 + y_1}{4}, \quad c_2 = \frac{3y_0 - y_1}{4}.$$

The solution of the general initial-value problem is thereby

$$y(t) = \frac{y_0 + y_1}{4} e^{3t} + \frac{3y_0 - y_1}{4} e^{-t} = \frac{e^{3t} + 3e^{-t}}{4} y_0 + \frac{e^{3t} - e^{-t}}{4} y_1.$$

Therefore the associated natural fundamental set of solutions is

$$N_0(t) = \frac{e^{3t} + 3e^{-t}}{4}, \quad N_1(t) = \frac{e^{3t} - e^{-t}}{4}.$$

**Solution (a) by Eigen Methods.** Because

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix},$$

the characteristic polynomial of  $\mathbf{A}$  is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z + 1)(z - 3).$$

The eigenvalues of  $\mathbf{A}$  are the roots of this polynomial, which are  $-1$  and  $3$ . Because

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix},$$

we can read off that  $\mathbf{A}$  has the eigenpairs

$$\left(-1, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

Set

$$\mathbf{V} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Because  $\det(\mathbf{V}) = 4$ , we see that

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & -2e^{-t} \\ e^{3t} & 2e^{3t} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{-t} + 2e^{3t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{-t} + 2e^{3t} \end{pmatrix}. \end{aligned}$$

**Solution (b) by Two-by-Two Formula.** Because

$$\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix},$$

the characteristic polynomial of  $\mathbf{A}$  is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z - 4)^2.$$

This is a perfect square with  $\mu = 4$ . Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{4t} \left[ \mathbf{I} + t(\mathbf{A} - 4\mathbf{I}) \right] = e^{4t} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \right] \\ &= e^{4t} \begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix}. \end{aligned}$$

**Solution (b) by the Natural Fundamental Set Method.** Because

$$\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix},$$

the characteristic polynomial of  $\mathbf{A}$  is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z - 4)^4.$$

Below we show that the natural fundamental set of solutions for  $t = 0$  associated with  $p(D) = D^2 - 8D + 16$  is

$$N_0(t) = (1 - 4t)e^{4t}, \quad N_1(t) = te^{4t}.$$

Then

$$\begin{aligned} e^{t\mathbf{A}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{A} = (1 - 4t)e^{4t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + te^{4t} \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix} \\ &= e^{4t} \begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix}. \end{aligned}$$

**From the Green Function.** Because  $p(s) = s^2 - 8s + 16 = (s - 4)^2$ , the Green function associated with  $p(D) = D^2 - 8D + 16$  is

$$\begin{aligned} g(t) &= \mathcal{L}^{-1} \left[ \frac{1}{p(s)} \right] (t) = \mathcal{L}^{-1} \left[ \frac{1}{s^2 - 8s + 16} \right] (t) \\ &= \mathcal{L}^{-1} \left[ \frac{1}{(s - 4)^2} \right] (t) = te^{4t}. \end{aligned}$$

Then, because the characteristic polynomial is  $p(s) = s^2 - 8s + 16$ , the natural fundamental set is

$$\begin{aligned} N_1(t) &= g(t) = te^{4t}, \\ N_0(t) &= N_1'(t) - 8g(t) = (e^{4t} + 4te^{4t}) - 8te^{4t} = e^{4t} - 4te^{4t}. \end{aligned}$$

**From the General Initial-Value Problem.** The general initial-value problem associated with  $p(D) = D^2 - 8D + 16$  is

$$y'' - 8y' + 16y = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

This has the general solution  $y(t) = c_1e^{4t} + c_2te^{4t}$ . Because

$$y'(t) = 4c_1e^{4t} + 4c_2te^{4t} + c_2e^{4t},$$

the general initial conditions yield

$$y_0 = y(0) = c_1, \quad y_1 = y'(0) = 4c_1 + c_2.$$

This system can be solved to obtain  $c_1 = y_0$  and  $c_2 = y_1 - 4y_0$ . The solution of the general initial-value problem is thereby

$$y(t) = y_0e^{4t} + (y_1 - 4y_0)te^{4t} = (1 - 4t)e^{4t}y_0 + te^{4t}y_1.$$

Therefore the associated natural fundamental set of solutions is

$$N_0(t) = (1 - 4t)e^{4t}, \quad N_1(t) = te^{4t}.$$

(12) Give the Green matrix for the system  $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$  when

(a)  $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

$$(b) \mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$$

**Solution (a).** By the solution to part (a) of the previous problem

$$e^{t\mathbf{A}} = \frac{1}{4} \begin{pmatrix} 2e^{3t} + 2e^{-t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}.$$

Therefore the Green matrix  $\mathbf{G}(t, s)$  is given by

$$\mathbf{G}(t, s) = e^{t\mathbf{A}} e^{-s\mathbf{A}} = e^{(t-s)\mathbf{A}} = \frac{1}{4} \begin{pmatrix} 2e^{3(t-s)} + 2e^{-(t-s)} & 4e^{3(t-s)} - 4e^{-(t-s)} \\ e^{3(t-s)} - e^{-(t-s)} & 2e^{3(t-s)} + 2e^{-(t-s)} \end{pmatrix}.$$

**Solution (b).** By the solution to part (b) of the previous problem

$$e^{t\mathbf{A}} = e^{4t} \begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix}.$$

Therefore the Green matrix  $\mathbf{G}(t, s)$  is given by

$$\mathbf{G}(t, s) = e^{t\mathbf{A}} e^{-s\mathbf{A}} = e^{(t-s)\mathbf{A}} = e^{4(t-s)} \begin{pmatrix} 1 + 2(t-s) & 4(t-s) \\ -(t-s) & 1 - 2(t-s) \end{pmatrix}.$$

(13) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix}.$$

Compute  $e^{t\mathbf{A}}$  given that the characteristic polynomial of  $\mathbf{A}$  is  $p(z) = z^3 + 9z$  and that the natural fundamental set of solutions associated with  $t = 0$  for the operator  $D^3 + 9D$  is

$$N_0(t) = 1, \quad N_1(t) = \frac{1}{3} \sin(3t), \quad N_2(t) = \frac{1}{9}(1 - \cos(3t)).$$

**Solution.** The natural fundamental set method says that

$$e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A} + N_2(t)\mathbf{A}^2.$$

Because  $N_0(t) = 1$ ,  $N_1(t) = \frac{1}{3} \sin(3t)$ ,  $N_2(t) = \frac{1}{9}(1 - \cos(3t))$ , and

$$\begin{aligned} \mathbf{A}^2 &= \begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 8 - 2 & 2 & -1 + 4 + 1 \\ -4 + 0 + 4 & -8 & 4 - 0 - 2 \\ 2 + 0 - 2 & 4 & -2 - 0 + 1 \end{pmatrix} = \begin{pmatrix} -9 & 2 & 4 \\ 0 & -8 & 2 \\ 0 & 4 & -1 \end{pmatrix}, \end{aligned}$$

we see that

$$\begin{aligned} e^{t\mathbf{A}} &= 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{3} \sin(3t) \begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix} + \frac{1}{9} (1 - \cos(3t)) \begin{pmatrix} -9 & 2 & 4 \\ 0 & -8 & 2 \\ 0 & 4 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(3t) - \frac{1}{3} \sin(3t) & -\frac{2}{3} \sin(3t) + \frac{2}{3} - \frac{2}{3} \cos(3t) & \frac{1}{3} \sin(3t) + \frac{4}{9} - \frac{4}{9} \cos(3t) \\ \frac{4}{3} \sin(3t) & \frac{1}{9} + \frac{8}{9} \cos(3t) & -\frac{2}{3} \sin(3t) + \frac{2}{9} - \frac{2}{9} \cos(3t) \\ -\frac{2}{3} \sin(3t) & \frac{4}{9} - \frac{4}{9} \cos(3t) & \frac{8}{9} + \frac{1}{9} \cos(3t) + \frac{1}{3} \sin(3t) \end{pmatrix}. \end{aligned}$$

(14) Solve each of the following initial-value problems.

$$(a) \quad \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$(b) \quad \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

**Solution (a).** The characteristic polynomial of  $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$  is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - z - 12 = (z - \frac{1}{2})^2 - (\frac{7}{2})^2.$$

This is a difference of squares with  $\mu = \frac{1}{2}$  and  $\nu = \frac{7}{2}$ . Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{\frac{1}{2}t} \left[ \cosh\left(\frac{7}{2}t\right) \mathbf{I} + \frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}} (\mathbf{A} - \frac{1}{2}\mathbf{I}) \right] \\ &= e^{\frac{1}{2}t} \left[ \cosh\left(\frac{7}{2}t\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}} \begin{pmatrix} \frac{3}{2} & 2 \\ 5 & -\frac{3}{2} \end{pmatrix} \right] \\ &= e^{\frac{1}{2}t} \begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7} \sinh\left(\frac{7}{2}t\right) & \frac{4}{7} \sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7} \sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7} \sinh\left(\frac{7}{2}t\right) \end{pmatrix}. \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\frac{1}{2}t} \begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7} \sinh\left(\frac{7}{2}t\right) & \frac{4}{7} \sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7} \sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7} \sinh\left(\frac{7}{2}t\right) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\frac{1}{2}t} \begin{pmatrix} \cosh\left(\frac{7}{2}t\right) - \frac{1}{7} \sinh\left(\frac{7}{2}t\right) \\ -\cosh\left(\frac{7}{2}t\right) + \frac{13}{7} \sinh\left(\frac{7}{2}t\right) \end{pmatrix}. \end{aligned}$$

**Solution (b).** The characteristic polynomial of  $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$  is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 5 = (z - 1)^2 + 2^2.$$



This is a sum of squares with  $\mu = 1$  and  $\nu = 2$ . Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[ \cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[ \cos(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix}. \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^t \begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) \\ -2\sin(2t) + \cos(2t) \end{pmatrix}. \end{aligned}$$

**Remark.** We could have used other methods to compute  $e^{t\mathbf{A}}$  in each part of the above problem. Alternatively, we could have constructed a fundamental matrix  $\Psi(t)$  and then determined  $\mathbf{c}$  so that  $\Psi(t)\mathbf{c}$  satisfies the initial conditions.

(15) Find a general solution for each of the following systems.

$$(a) \quad \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(b) \quad \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(c) \quad \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

**Solution (a).** We must find a general solution for the system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic polynomial of  $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$  is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 1 = (z - 1)^2.$$

This is a perfect square with  $\mu = 2$ . Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t [\mathbf{I} + t(\mathbf{A} - \mathbf{I})] = e^t \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} 1 + 2t & -4t \\ t & 1 - 2t \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^t \begin{pmatrix} 1+2t \\ t \end{pmatrix} + c_2 e^t \begin{pmatrix} -4t \\ 1-2t \end{pmatrix}. \end{aligned}$$

**Solution (b) by Two-by-Two Formula.** We must find a general solution for the system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic polynomial of  $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$  is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4^2.$$

This is a sum of squares with  $\mu = 0$  and  $\nu = 4$ . Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= \left[ \cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}\mathbf{A} \right] = \left[ \cos(4t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4} \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \right] \\ &= \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ \sin(4t) \end{pmatrix} + c_2 \begin{pmatrix} -\frac{5}{4}\sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

**Solution (b) by Eigen Methods.** We must find a general solution for the system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic polynomial of  $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$  is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16.$$

The eigenvalues of  $\mathbf{A}$  are the roots of this polynomial, which are  $\pm i4$ . Because

$$\mathbf{A} - i4\mathbf{I} = \begin{pmatrix} 2 - i4 & -5 \\ 4 & -2 - i4 \end{pmatrix}, \quad \mathbf{A} + i4\mathbf{I} = \begin{pmatrix} 2 + i4 & -5 \\ 4 & -2 + i4 \end{pmatrix},$$

we can read off that  $\mathbf{A}$  has the eigenpairs

$$\left( i4, \begin{pmatrix} 1+i2 \\ 2 \end{pmatrix} \right), \quad \left( -i4, \begin{pmatrix} 1-i2 \\ 2 \end{pmatrix} \right).$$

Therefore the system has the complex-valued solution

$$\begin{aligned} e^{i4t} \begin{pmatrix} 1+i2 \\ 2 \end{pmatrix} &= (\cos(4t) + i\sin(4t)) \begin{pmatrix} 1+i2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(4t) - 2\sin(4t) + i2\cos(4t) + i\sin(4t) \\ 2\cos(4t) + i2\sin(4t) \end{pmatrix}. \end{aligned}$$

By taking real and imaginary parts, we obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re} \left( e^{i4t} \begin{pmatrix} 1+i2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix}, \\ \mathbf{x}_2(t) &= \operatorname{Im} \left( e^{i4t} \begin{pmatrix} 1+i2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix} + c_2 \begin{pmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{pmatrix}.$$

**Solution (c) by Two-by-Two Formula.** We must find a general solution for the system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic polynomial of  $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$  is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z - 3)^2 + 4^2.$$

This is a sum of squares with  $\mu = 3$  and  $\nu = 4$ . Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t} \left[ \cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}(\mathbf{A} - 3\mathbf{I}) \right] \\ &= e^{3t} \left[ \cos(4t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4} \begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix} \right] \\ &= e^{3t} \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{3t} \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{3t} \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ -\frac{5}{4}\sin(4t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

**Solution (c) by Eigen Methods.** We must find a general solution for the system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic polynomial of  $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$  is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z - 3)^2 + 16.$$

The eigenvalues of  $\mathbf{A}$  are the roots of this polynomial, which are  $3 \pm i4$ . Because

$$\mathbf{A} - (3 + i4)\mathbf{I} = \begin{pmatrix} 2 - i4 & 4 \\ -5 & -2 - i4 \end{pmatrix}, \quad \mathbf{A} - (3 - i4)\mathbf{I} = \begin{pmatrix} 2 + i4 & 4 \\ -5 & -2 + i4 \end{pmatrix},$$

we can read off that  $\mathbf{A}$  has the eigenpairs

$$\left( 3 + i4, \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} \right), \quad \left( 3 - i4, \begin{pmatrix} -2 \\ 1 + i2 \end{pmatrix} \right).$$

Therefore the system has the complex-valued solution

$$\begin{aligned} e^{(3+i4)t} \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} &= e^{3t} (\cos(4t) + i \sin(4t)) \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} \\ &= e^{3t} \begin{pmatrix} -2 \cos(4t) - i2 \sin(4t) \\ \cos(4t) + 2 \sin(4t) + i \sin(4t) - i2 \cos(4t) \end{pmatrix}. \end{aligned}$$

By taking real and imaginary parts, we obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re} \left( e^{(3+i4)t} \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} \right) = e^{3t} \begin{pmatrix} -2 \cos(4t) \\ \cos(4t) + 2 \sin(4t) \end{pmatrix}, \\ \mathbf{x}_2(t) &= \operatorname{Im} \left( e^{(3+i4)t} \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} \right) = e^{3t} \begin{pmatrix} -2 \sin(4t) \\ \sin(4t) - 2 \cos(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} -2 \cos(4t) \\ \cos(4t) + 2 \sin(4t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -2 \sin(4t) \\ \sin(4t) - 2 \cos(4t) \end{pmatrix}.$$

(16) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of  $\mathbf{A}$  associated with 1.

**Solution.** The eigenvectors of  $\mathbf{A}$  associated with 1 are all nonzero vectors  $\mathbf{v}$  that satisfy  $\mathbf{A}\mathbf{v} = \mathbf{v}$ . Equivalently, they are all nonzero vectors  $\mathbf{v}$  that satisfy  $(\mathbf{A} - \mathbf{I})\mathbf{v} = \mathbf{0}$ , which is

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}.$$

The entries of  $\mathbf{v}$  thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} v_1 - v_2 + v_3 &= 0, \\ v_1 - v_3 &= 0, \\ -v_2 + 2v_3 &= 0. \end{aligned}$$

We may solve this system either by elimination or by row reduction. By either method we find that its general solution is

$$v_1 = \alpha, \quad v_2 = 2\alpha, \quad v_3 = \alpha, \quad \text{for any constant } \alpha.$$

The eigenvectors of  $\mathbf{A}$  associated with 1 thereby have the form

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{for any nonzero constant } \alpha.$$

(17) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}.$$

- (a) Find all the eigenvalues of  $\mathbf{A}$ .
- (b) For each eigenvalue of  $\mathbf{A}$  find all of its eigenvectors.
- (c) Diagonalize  $\mathbf{A}$ .
- (d) Compute  $e^{t\mathbf{A}}$ .
- (e) Compute  $(s\mathbf{I} - \mathbf{A})^{-1}$  for every  $s$  where it is defined.

**Solution (a).** The characteristic polynomial of  $\mathbf{A}$  is given by

$$p(z) = z^2 - \text{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 15 = (z - 1)^2 - 16.$$

The eigenvalues of  $\mathbf{A}$  are the roots of this polynomial, which are  $1 \pm 4$ , or simply  $-3$  and  $5$ .

**Solution (b) by the Cayley-Hamilton Method.** We have

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}, \quad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}.$$

Every nonzero column of  $\mathbf{A} - 5\mathbf{I}$  has the form

$$\alpha_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0.$$

These are all the eigenvectors associated with  $-3$ . Similarly, every nonzero column of  $\mathbf{A} + 3\mathbf{I}$  has the form

$$\alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0.$$

These are all the eigenvectors associated with  $5$ .

**Solution (c).** If we use the eigenpairs

$$\left(-3, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right),$$

then set

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}.$$

Because  $\det(\mathbf{V}) = 1 \cdot 2 - (-2) \cdot 3 = 2 + 6 = 8$ , we see that

$$\mathbf{V}^{-1} = \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}.$$

We conclude that  $\mathbf{A}$  has the diagonalization

$$\mathbf{A} = \mathbf{VDV}^{-1} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}.$$

You do not have to multiply these matrices out. Had we started with different eigenpairs, the steps would be the same as above but we would obtain a different diagonalization.

**Solution (d).** Because  $\mathbf{A} = \mathbf{VDV}^{-1}$  by part (c), we have

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{5t} \end{pmatrix} \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2e^{-3t} & -3e^{-3t} \\ 2e^{5t} & e^{5t} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 2e^{-3t} + 6e^{5t} & -3e^{-3t} + 3e^{5t} \\ -4e^{-3t} + 4e^{5t} & 6e^{-3t} + 2e^{5t} \end{pmatrix}. \end{aligned}$$

**Solution (e).** Because  $\mathbf{A} = \mathbf{VDV}^{-1}$  by part (c), we have

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})^{-1} &= \mathbf{V}(s\mathbf{I} - \mathbf{D})^{-1}\mathbf{V}^{-1} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s-5} \end{pmatrix} \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{s+3} & \frac{-3}{s+3} \\ \frac{2}{s-5} & \frac{1}{s-5} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} \frac{2}{s+3} + \frac{6}{s-5} & \frac{-3}{s+3} + \frac{3}{s-5} \\ \frac{-4}{s+3} + \frac{4}{s-5} & \frac{2}{s+3} + \frac{2}{s-5} \end{pmatrix}. \end{aligned}$$

This is defined for every  $s$  except at  $s = -3$  and  $s = 5$ .

(18) What answer will be produced by the following Matlab command?

```
>> A = [1 4; 3 2]; [vect, val] = eig(sym(A))
```

You do not have to give the answer in Matlab format.

**Solution.** The Matlab command will produce the eigenpairs of  $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ . The characteristic polynomial of  $\mathbf{A}$  is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 3z - 10 = (z - 5)(z + 2),$$

so its eigenvalues are 5 and  $-2$ . Because

$$\mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}, \quad \mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix},$$

we can read off that the eigenpairs are

$$\left( 5, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \quad \left( -2, \begin{pmatrix} -4 \\ 3 \end{pmatrix} \right).$$

(19) A  $3 \times 3$  matrix  $\mathbf{A}$  has the eigenpairs

$$\left(-3, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right).$$

- (a) Give an invertible matrix  $\mathbf{V}$  and a diagonal matrix  $\mathbf{D}$  such that  $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$ .  
 (You do not have to compute either  $\mathbf{V}^{-1}$  or  $e^{t\mathbf{A}}$ !)  
 (b) Give a fundamental matrix for the system  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ .

**Solution (a).** One choice for  $\mathbf{V}$  and  $\mathbf{D}$  is

$$\mathbf{V} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

**Solution (b).** Use the given eigenpairs to construct the special solutions

$$\mathbf{x}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3(t) = e^{5t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$

Then a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = (\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)) = \begin{pmatrix} e^{-3t} & -e^{2t} & e^{5t} \\ e^{-3t} & e^{2t} & -e^{5t} \\ 0 & e^{2t} & 2e^{5t} \end{pmatrix}.$$

**Alternative Solution (b).** Given the  $\mathbf{V}$  and  $\mathbf{D}$  from part (a), a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix} = \begin{pmatrix} e^{-3t} & -e^{2t} & e^{5t} \\ e^{-3t} & e^{2t} & -e^{5t} \\ 0 & e^{2t} & 2e^{5t} \end{pmatrix}.$$

### Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \quad \text{for } s > a.$$

$$\mathcal{L}[e^{at} \cos(bt)](s) = \frac{s-a}{(s-a)^2 + b^2} \quad \text{for } s > a.$$

$$\mathcal{L}[e^{at} \sin(bt)](s) = \frac{b}{(s-a)^2 + b^2} \quad \text{for } s > a.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \quad \text{where } J(s) = \mathcal{L}[j(t)](s).$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \quad \text{where } J(s) = \mathcal{L}[j(t)](s).$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \quad \text{where } J(s) = \mathcal{L}[j(t)](s) \text{ and } u \text{ is the unit step function.}$$

$$\mathcal{L}[\delta(t-c)h(t)](s) = e^{-cs}h(c) \quad \text{where } \delta \text{ is the unit impulse.}$$