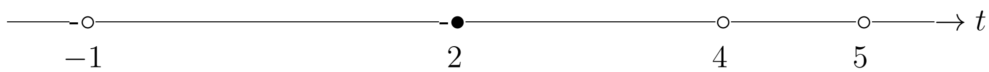
Solutions of the Sample Problems for the Second In-Class Exam
Math 246, Spring 2018, Professor David Levermore

(1) Give the interval of definition for the solution of the initial-value problem

$$x''' + \frac{\cos(3t)}{4-t}\,x' + \frac{\sin(2t)}{5-t}\,x = \frac{e^{-2t}}{1+t}\,, \qquad x(2) = x'(2) = x''(2) = 0\,.$$

Solution. The equation is linear and is already in normal form. Notice the following.
◇ The coefficient of x' is undefined at $t = 4$ and is continuous elsewhere.
◇ The coefficient of x is undefined at $t = 5$ and is continuous elsewhere.
◇ The forcing is undefined at $t = -1$ and is continuous elsewhere.
Plotting these points along with the inital time $t = 2$ on a time-line gives



Therefore the interval of definition is $(-1, 4)$ because:
- the initial time $t = 2$ is in $(-1, 4)$;
- all the coefficients and the forcing are continuous over $(-1, 4)$;
- the forcing is undefined at $t = -1$;
- the coefficient of x' is undefined at $t = 4$.

Remark. All four reasons must be given for full credit.
◦ The first two reasons are why a (unique) solution exists over the interval $(-1, 4)$.
◦ The last two reasons are why this solution does not exist over a larger interval.

(2) Suppose that $Z_1(t)$, $Z_2(t)$, and $Z_3(t)$ are solutions of the differential equation

$$z''' + 2z'' + (1 + t^2)z = 0\,.$$

Suppose we know that $\mathrm{Wr}[Z_1, Z_2, Z_3](1) = 5$. What is $\mathrm{Wr}[Z_1, Z_2, Z_3](t)$?

Solution. The Abel Theorem says that $w = \mathrm{Wr}[Z_1, Z_2, Z_3](t)$ satisfies $w' + 2w = 0$. It follows that $w(t) = ce^{-2t}$ for some c. Because $w(1) = \mathrm{Wr}[Z_1, Z_2, Z_3](1) = 5$, this initial condition implies that $ce^{-2\cdot 1} = 5$, whereby $c = 5e^2$. Therefore $w(t) = 5e^2e^{-2t}$, which means that

$$\mathrm{Wr}[Z_1, Z_2, Z_3](t) = 5e^2e^{-2t}\,.$$

(3) Show that the functions $X_1(t) = 1$, $X_2(t) = \cos(t)$, and $X_3(t) = \sin(t)$ are linearly independent.

Solution. The Wronskian of these functions is

$$\begin{aligned}\mathrm{Wr}[X_1, X_2, X_3](t) &= \det\begin{pmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{pmatrix} \\ &= 1 \cdot (-\sin(t)) \cdot (-\sin(t)) - (-\cos(t)) \cdot \cos(t) \cdot 1 \\ &= \sin(t)^2 + \cos(t)^2 = 1\,.\end{aligned}$$

Because $\mathrm{Wr}[X_1, X_2, X_3](t) \neq 0$, the functions are linearly independent.

Alternative Solution. Suppose that

$$0 = c_1X_1(t) + c_2X_2(t) + c_3X_3(t) = c_1 + c_2\cos(t) + c_3\sin(t)\,.$$

To show linear independence we must show that $c_1 = c_2 = c_3 = 0$. But setting $t = 0$, $t = \pi$, and $t = \frac{\pi}{2}$ into this relation yields the linear algebraic system

$$\begin{aligned} 0 &= c_1 + c_2\cos(0) + c_3\sin(0) = c_1 + c_2\,,\\ 0 &= c_1 + c_2\cos(\pi) + c_2\sin(\pi) = c_1 - c_3\,,\\ 0 &= c_1 + c_2\cos(\tfrac{\pi}{2}) + c_3\sin(\tfrac{\pi}{2}) = c_1 + c_3\,. \end{aligned}$$

By adding the first two equations we see that $c_1 = 0$. By subtracting the second equation from the first we see that $c_2 = 0$. By plugging $c_1 = 0$ into the thrid equation we see that $c_3 = 0$, Therefore $c_1 = c_2 = c_3 = 0$, whereby $X_1(t)$, $X_2(t)$, and $X_3(t)$ are linearly independent.

(4) Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2 + i3$, $-2 - i3$, $i7$, $i7$, $-i7$, $-i7$, 5, 5, 5, -3, 0, 0.

(a) Give the order of L.

Solution. Because there are 12 roots listed, the degree of the characteristic polynomial must be 12, whereby the order of L is 12.

(b) Give a real general solution of the homogeneous equation $\mathrm{L}y = 0$.

Solution. A fundamental set of twelve real-valued solutions is built as follows.

◇ The conjugate pair of simple roots $-2 \pm i3$ contributes

$$e^{-2t}\cos(3t) \quad\text{and}\quad e^{-2t}\sin(3t)\,.$$

◇ The conjugate pair of double roots $\pm i7$ contributes

$$\cos(7t)\,,\quad \sin(7t)\,,\quad t\,\cos(7t)\,,\quad\text{and}\quad t\,\sin(7t)\,.$$

◇ The triple real root 5 contributes e^{5t}, $t\,e^{5t}$, and t^2e^{5t}.

◇ The simple real root -3 contributes e^{-3t}.

◇ The double real root 0 contributes 1 and t.

Therefore a real general solution is

$$\begin{aligned} y = {}& c_1e^{-2t}\cos(3t) + c_2e^{-2t}\sin(3t)\\ &+ c_3\cos(7t) + c_4\sin(7t) + c_5t\,\cos(7t) + c_6t\,\sin(7t)\\ &+ c_7e^{5t} + c_8t\,e^{5t} + c_9t^2e^{5t} + c_{10}e^{-3t} + c_{11} + c_{12}t\,. \end{aligned}$$

(5) Give the natural fundamental set of solutions associated with $t = 0$ for each of the following equations.

(a) $v'' - 6v' + 9v = 0$

Solution. The general initial-value problem associated with $t = 0$ is

$$v'' - 6v' + 9v = 0\,, \qquad v(0) = v_0\,, \quad v'(0) = v_1\,.$$

This is a *homogeneous* linear equation for $v(t)$ with *constant* coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 6z + 9 = (z-3)^2 \,.$$

This has the double real root 3, which yields a real general solution

$$v(t) = c_1 e^{3t} + c_2 t\, e^{3t} \,.$$

Because

$$v'(t) = 3c_1 e^{3t} + 3c_2 t\, e^{3t} + c_2 e^{3t} \,,$$

when the general initial conditions are imposed, we find

$$v(0) = c_1 = v_0 \,, \qquad v'(0) = 3c_1 + c_2 = v_1 \,.$$

These are solved to find $c_1 = v_0$ and $c_2 = v_1 - 3v_0$. So the solution of the general initial-value problem is

$$v(t) = v_0 e^{3t} + (v_1 - 3v_0) t\, e^{3t} = (1-3t)e^{3t} v_0 + t\, e^{3t} v_1 \,.$$

Therefore the natural fundamental set of solutions associated with $t = 0$ is

$$N_0(t) = (1-3t)e^{3t} \,, \qquad N_1(t) = t\, e^{3t} \,.$$

(b) $\ddot{y} + 4\dot{y} + 20y = 0$

Solution. The general initial-value problem associated with $t = 0$ is

$$\ddot{y} + 4\dot{y} + 20y = 0 \,, \qquad y(0) = y_0 \,, \quad \dot{y}(0) = y_1 \,.$$

This is a *homogeneous* linear equation for $y(t)$ with *constant* coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 20 = (z+2)^2 + 4^2 \,.$$

This has the conjugate pair of roots $-2 \pm 4i$, which yields a real general solution

$$y(t) = c_1 e^{-2t} \cos(4t) + c_2 e^{-2t} \sin(4t) \,.$$

Because

$$\dot{y}(t) = -2c_1 e^{-2t} \cos(4t) - 4c_1 e^{-2t} \sin(4t) - 2c_2 e^{-2t} \sin(4t) + 4c_2 e^{-2t} \cos(4t) \,.$$

when the general initial conditions are imposed, we find

$$y(0) = c_1 = y_0 \,, \qquad \dot{y}(0) = -2c_1 + 4c_2 = y_1 \,.$$

These are solved to find $c_1 = y_0$ and $c_2 = \frac{1}{2}y_0 + \frac{1}{4}y_1$. So the solution of the general initial-value problem is

$$\begin{aligned} y(t) &= y_0 e^{-2t} \cos(4t) + \left(\tfrac{1}{2}y_0 + \tfrac{1}{4}y_1\right) e^{-2t} \sin(4t) \\ &= e^{-2t}\big(\cos(4t) + \tfrac{1}{2}\sin(4t)\big)\, y_0 + e^{-2t}\tfrac{1}{4}\sin(4t)\, y_1 \,. \end{aligned}$$

Therefore the natural fundamental set of solutions associated with $t = 0$ is

$$N_0(t) = e^{-2t}\big(\cos(4t) + \tfrac{1}{2}\sin(4t)\big) \,, \qquad N_1(t) = e^{-2t}\tfrac{1}{4}\sin(4t) \,.$$

(6) Let $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Solve each of the following initial-value problems.

(a) $\mathrm{D}^2 y + 4\mathrm{D}y + 4y = 0\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,.$

Solution. This is a *homogeneous* linear equation for $y(t)$ with *constant* coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 4 = (z+2)^2\,.$$

This has the double real root -2, which yields a real general solution given by

$$y(t) = c_1 e^{-2t} + c_2 t\, e^{-2t}\,.$$

Because

$$y'(t) = -2c_1 e^{-2t} - 2c_2 t\, e^{-2t} + c_2 e^{-2t}\,,$$

when the initial conditions are imposed, we find that

$$y(0) = c_1 = 1\,, \qquad y'(0) = -2c_1 + c_2 = 0\,.$$

These are solved to find $c_1 = 1$ and $c_2 = 2$. Therefore the solution of the initial-value problem is

$$y(t) = e^{-2t} + 2t\, e^{-2t} = (1+2t)e^{-2t}\,.$$

(b) $\mathrm{D}^2 w + 9w = 20e^t\,, \qquad w(0) = 0\,, \quad w'(0) = 0\,.$

Solution. This is a *nonhomogeneous* linear equation for $w(t)$ with *constant* coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2\,.$$

This has the conjugate pair of simple roots $\pm i3$, which yields a real general solution of the associated homogeneous problem given by

$$w_H(t) = c_1 \cos(3t) + c_2 \sin(3t)\,.$$

The forcing $20e^t$ has degree $d = 0$, characteristic $\mu + i\nu = 1$, and multiplicity $m = 0$. A particular solution $w_P(t)$ can be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients.

Below we show that each of these methods gives $w_P(t) = 2e^t$, which yields a real general solution

$$w(t) = c_1 \cos(3t) + c_2 \sin(3t) + 2e^t\,.$$

Because

$$w'(t) = -3c_1 \sin(3t) + 3c_2 \cos(3t) + 2e^t\,,$$

when the initial conditions are imposed we find that

$$w(0) = c_1 + 2 = 0\,, \qquad w'(0) = 3c_2 + 2 = 0\,.$$

These are solved to obtain $c_1 = -2$ and $c_2 = -\frac{2}{3}$. Therefore the solution of the initial-value problem is

$$w(t) = -2\cos(3t) - \tfrac{2}{3}\sin(3t) + 2e^t\,.$$

Key Indentity Evaluations. Because $d = m = 0$, we can simply evaluate the Key Identity at $z = \mu + i\nu = 1$, to find

$$\mathrm{L}(e^t) = p(1)e^t = (1^2 + 9)e^t = 10e^t\,.$$

Multiplying this equation by 2 yields $\mathrm{L}(2e^t) = 20e^t$. Hence, $w_P(t) = 2e^t$.

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big)\,,$$

this formula gives the particular solution

$$w_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right)\,.$$

For this problem $f(t) = 20e^t$ and $p(z) = z^2 + 9$, so that $\mu = 1$, $\nu = 0$, $\alpha = 20$, $\beta = 0$, and $m = 0$, whereby

$$w_P(t) = e^t\,\frac{20}{p(1)} = \frac{20}{1^2 + 9}\,e^t = \frac{20}{10}\,e^t = 2e^t\,.$$

Undetermined Coefficients. Because $m + d = 0$, $m = 0$, and $\mu + i\nu = 1$, there is a particular solution in the form

$$w_P(t) = Ae^t\,.$$

Because $w_P'(t) = Ae^t$ and $w_P''(t) = Ae^t$, we see that

$$\mathrm{L}w_P(t) = w_P''(t) + 9w_P(t) = Ae^t + 9Ae^t = 10Ae^t\,.$$

By setting $\mathrm{L}w_P(t) = 10Ae^t = 20e^t$, we see that $A = 2$. Hence, $w_P(t) = 2e^t$.

(7) Give a real general solution for each of the following equations.

(a) $\ddot{u} + 4\dot{u} + 5u = 3\cos(2t)$

Solution. This is a *nonhomogeneous* linear equation for $u(t)$ with *constant* coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 5 = (z + 2)^2 + 1\,.$$

This has the conjugate pair of simple roots $-2 \pm i$, which yields a real general solution of the associated homogeneous problem given by

$$u_H(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)\,.$$

The forcing $3\cos(2t)$ has degree $d = 0$, characteristic $\mu + i\nu = i2$, and multiplicity $m = 0$. A particular solution $u_P(t)$ can be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients.

Below we show that each of these methods gives the particular solution

$$u_P(t) = \tfrac{3}{65}\cos(2t) + \tfrac{24}{65}\sin(2t)\,.$$

Therefore a real general solution is

$$u = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t) + \tfrac{3}{65}\cos(2t) + \tfrac{24}{65}\sin(2t)\,.$$

Key Indentity Evaluations. Because $d = m = 0$, we can simply evaluate the Key Identity at $z = \mu + i\nu = i2$, to find

$$\mathrm{L}\big(e^{i2t}\big) = p(i2)e^{i2t} = \big((i2)^2 + 4(i2) + 5\big)e^{i2t} = (1 + i8)e^{i2t}\,.$$

Because the forcing $3\cos(2t) = 3\,\mathrm{Re}\big(e^{i2t}\big)$, we divide the above by $1 + i8$ and multiply by 3 to find

$$\mathrm{L}\left(\frac{3}{1+i8}\,e^{i2t}\right) = 3e^{i2t}\,.$$

Hence,

$$\begin{aligned} u_P(t) &= \mathrm{Re}\left(\frac{3}{1+i8}\,e^{i2t}\right) = \mathrm{Re}\left(\frac{3(1-i8)}{1^2+8^2}\,e^{i2t}\right) = \tfrac{3}{65}\,\mathrm{Re}\big((1-i8)e^{i2t}\big) \\ &= \tfrac{3}{65}\big(\cos(2t) + 8\sin(2t)\big) = \tfrac{3}{65}\cos(2t) + \tfrac{24}{65}\sin(2t)\,. \end{aligned}$$

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big)\,,$$

this formula gives the particular solution

$$u_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right)\,.$$

For this problem $f(t) = 3\cos(2t)$ and $p(z) = z^2 + 4z + 5$, so that $\mu = 0$, $\nu = 2$, $\alpha = 3$, $\beta = 0$, $m = 0$, and $p(i2) = (i2)^2 + 4\cdot i2 + 5 = -4 + i8 + 5 = 1 + i8$, whereby

$$\begin{aligned} u_P(t) &= \mathrm{Re}\left(\frac{3}{1+i8}\,e^{i2t}\right) = \mathrm{Re}\left(\frac{3(1-i8)}{1^2+8^2}\,e^{i2t}\right) = \tfrac{3}{65}\,\mathrm{Re}\big((1-i8)e^{i2t}\big) \\ &= \tfrac{3}{65}\big(\cos(2t) + 8\sin(2t)\big) = \tfrac{3}{65}\cos(2t) + \tfrac{24}{65}\sin(2t)\,. \end{aligned}$$

Undetermined Coefficients. Because $m + d = 0$, $m = 0$, and $\mu + i\nu = i2$, there is a particular solution in the form

$$u_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$\begin{aligned} \dot{u}_P(t) &= -2A\sin(2t) + 2B\cos(2t)\,, \\ \ddot{u}_P(t) &= -4A\cos(2t) - 4B\sin(2t)\,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}u_P(t) &= \ddot{u}_P(t) + 4\dot{u}_P(t) + 5u_P(t) \\ &= \big[-4A\cos(2t) - 4B\sin(2t)\big] + 4\big[-2A\sin(2t) + 2B\cos(2t)\big] \\ &\quad + 5\big[A\cos(2t) + B\sin(2t)\big] \\ &= (A + 8B)\cos(2t) + (B - 8A)\sin(2t)\,. \end{aligned}$$

By setting $\mathrm{L}u_P(t) = (A + 8B)\cos(2t) + (B - 8A)\sin(2t) = 3\cos(2t)$, we see that

$$A + 8B = 3\,, \qquad B - 8A = 0\,.$$

We find that $A = \frac{3}{65}$ and $B = \frac{24}{65}$. Hence, $u_P(t) = \frac{3}{65}\cos(2t) + \frac{24}{65}\sin(2t)$.

(b) $x'' - x = t\,e^t$

Solution. This is a *nonhomogeneous* linear equation for $x(t)$ with *constant* coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 1 = (z+1)(z-1)\,.$$

This has the simple real roots -1 and 1, which yields a real general solution of the associated homogeneous problem given by

$$x_H(t) = c_1 e^{-t} + c_2 e^t\,.$$

The forcing $t\,e^t$ has degree $d = 1$, characteristic $\mu + i\nu = 1$, and multiplicity $m = 1$. A particular solution $x_P(t)$ can be found by using either Key Identity Evaluations or Undetermined Coefficients.

Below we show that each of these methods gives the particular solution

$$x_P(t) = \tfrac{1}{4}(t^2 - t)\,e^t\,.$$

Therefore a real general solution is

$$x = c_1 e^{-t} + c_2 e^t + \tfrac{1}{4}(t^2 - t)\,e^t\,.$$

Key Indentity Evaluations. Because $m + d = 2$, we need the Key Identity and its first two derivatives with respect to z:

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= (z^2 - 1)e^{zt}\,,\\ \mathrm{L}\big(t\,e^{zt}\big) &= (z^2 - 1)t\,e^{zt} + 2ze^{zt}\,,\\ \mathrm{L}\big(t^2e^{zt}\big) &= (z^2 - 1)t^2e^{zt} + 4zt\,e^{zt} + 2e^{zt}\,. \end{aligned}$$

Evaluate these at $z = \mu + i\nu = 1$ to find

$$\mathrm{L}\big(e^t\big) = 0\,, \qquad \mathrm{L}\big(t\,e^t\big) = 2e^t\,, \qquad \mathrm{L}\big(t^2e^t\big) = 4t\,e^t + 2e^t\,.$$

Subtracting the second equation from the third yields

$$\mathrm{L}\big(t^2e^t - t\,e^t\big) = 4t\,e^t\,.$$

Dividing this equation by 4 gives $\mathrm{L}\big(\frac{1}{4}(t^2 - t)\,e^t\big) = t\,e^t$. Hence, $x_P(t) = \frac{1}{4}(t^2 - t)\,e^t$.

Undetermined Coefficients. Because $m + d = 2$, $m = 1$, and $\mu + i\nu = 1$, there is a particular solution in the form

$$x_P(t) = \big(A_0t^2 + A_1t\big)\,e^t\,,$$

Because

$$\begin{aligned} x_P'(t) &= \big(A_0t^2 + A_1t\big)\,e^t + \big(2A_0t + A_1\big)\,e^t\\ &= \big(A_0t^2 + (2A_0 + A_1)t + A_1\big)\,e^t\,,\\ x_P''(t) &= \big(A_0t^2 + (2A_0 + A_1)t + A_1\big)\,e^t + \big(2A_0t + (2A_0 + A_1)\big)\,e^t\\ &= \big(A_0t^2 + (4A_0 + A_1)t + 2A_0 + 2A_1\big)\,e^t\,, \end{aligned}$$

we see that

$$\begin{aligned}
\mathrm{L}x_P(t) &= x_P''(t) - x_P(t) \\
&= \big(A_0 t^2 + (4A_0 + A_1)t + 2A_0 + 2A_1\big)\, e^t - \big(A_0 t^2 + A_1 t\big)\, e^t \\
&= \big(4A_0 t + 2A_0 + 2A_1\big)\, e^t = 4A_0 t\, e^t + 2(A_0 + A_1)e^t \,.
\end{aligned}$$

By setting $\mathrm{L}x_P(t) = 4A_0 t\, e^t + 2(A_0 + A_1)e^t = t\, e^t$, we obtain $4A_0 = 1$ and $A_0 + A_1 = 0$. It follows that $A_0 = \frac{1}{4}$ and $A_1 = -\frac{1}{4}$. Hence, $x_P(t) = \frac{1}{4}(t^2 - t)\, e^t$.

(c) $y'' - y = \dfrac{1}{1 + e^t}$

Solution. This is a *nonhomogeneous* linear equation for $y(t)$ with *constant* coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 1 = (z - 1)(z + 1)\,.$$

This has the simple real roots 1 and -1, which yields a real general solution of the associated homogeneous problem given by

$$y_H(t) = c_1 e^t + c_2 e^{-t}\,.$$

The forcing does not have the characteristic form required for us to use either Key Identity Evaluations or Undertermined Coefficients. Therefore we must use either the Green Function method or the Variation of Parameters method.

Green Function. The Green function $g(t)$ satisfies

$$g'' - g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1 e^t + c_2 e^{-t}$. The first initial condition implies $g(0) = c_1 + c_2 = 0$. Because $g'(t) = c_1 e^t - c_2 e^{-t}$, the second initial condition yields $g'(0) = c_1 - c_2 = 1$. It follows that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$, whereby $g(t) = \frac{1}{2}(e^t - e^{-t})$.

Because the equation is in normal form, the Green Function method gives a particular solution by

$$\begin{aligned}
Y_P(t) = \int_0^t g(t - s)\, \frac{1}{1 + e^s}\, \mathrm{d}s &= \int_0^t \tfrac{1}{2}\big(e^{t-s} - e^{-t+s}\big)\, \frac{1}{1 + e^s}\, \mathrm{d}s \\
&= \tfrac{1}{2} e^t \int_0^t \frac{e^{-s}}{1 + e^s}\, \mathrm{d}s - \tfrac{1}{2} e^{-t} \int_0^t \frac{e^s}{1 + e^s}\, \mathrm{d}s\,.
\end{aligned}$$

The definite integrals on the right-hand side can be evaluated as

$$\begin{aligned}
\int_0^t \frac{e^{-s}}{1 + e^s}\, \mathrm{d}s &= \int_0^t \frac{e^{-2s}}{e^{-s} + 1}\, \mathrm{d}s = \int_0^t e^{-s} - \frac{e^{-s}}{e^{-s} + 1}\, \mathrm{d}s \\
&= \Big[- e^{-s} + \log\big(e^{-s} + 1\big)\Big]\Big|_{s=0}^{t} = 1 - e^{-t} + \log\!\left(\frac{e^{-t} + 1}{2}\right), \\
\int_0^t \frac{e^s}{1 + e^s}\, \mathrm{d}s &= \log\big(1 + e^s\big)\Big|_{s=0}^{t} = \log\!\left(\frac{1 + e^t}{2}\right).
\end{aligned}$$

Hence, a particular solution $Y_P(t)$ is given by

$$Y_P(t) = \tfrac{1}{2}\left[e^t - 1 + e^t \log\!\left(\frac{e^{-t} + 1}{2}\right)\right] - \tfrac{1}{2} e^{-t} \log\!\left(\frac{1 + e^t}{2}\right).$$

Therefore a real general solution is $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above. This yields

$$y = c_1 e^t + c_2 e^{-t} + \tfrac{1}{2}\left[e^t - 1 + e^t \log\left(\frac{e^{-t}+1}{2}\right)\right] - \tfrac{1}{2} e^{-t} \log\left(\frac{1+e^t}{2}\right).$$

Variation of Parameters. Seek a solution in the form

$$y = u_1(t)\, e^t + u_2(t)\, e^{-t}\,,$$

where $u_1(t)$ and $u_2(t)$ satisfy

$$\begin{aligned} u_1'(t)\, e^t + u_2'(t)\, e^{-t} &= 0\,, \\ u_1'(t)\, e^t - u_2'(t)\, e^{-t} &= \frac{1}{1+e^t}\,. \end{aligned}$$

Solve this system to obtain

$$u_1'(t) = \tfrac{1}{2}\frac{e^{-t}}{1+e^t}\,, \qquad u_2'(t) = -\tfrac{1}{2}\frac{e^t}{1+e^t}\,.$$

Integrate these equations to find

$$\begin{aligned} u_1(t) &= \tfrac{1}{2}\int \frac{e^{-t}}{1+e^t}\,\mathrm{d}t = \tfrac{1}{2}\int \frac{e^{-2t}}{e^{-t}+1}\,\mathrm{d}t \\ &= \tfrac{1}{2}\int e^{-t} - \frac{e^{-t}}{e^{-t}+1}\,\mathrm{d}t = -\tfrac{1}{2}e^{-t} + \tfrac{1}{2}\log\bigl(e^{-t}+1\bigr) + c_1\,, \\ u_2(t) &= -\tfrac{1}{2}\int \frac{e^t}{1+e^t}\,\mathrm{d}t = -\tfrac{1}{2}\log\bigl(1+e^t\bigr) + c_2\,. \end{aligned}$$

Therefore a real general solution is

$$y = c_1 e^t + c_2 e^{-t} - \tfrac{1}{2} + \tfrac{1}{2}e^t \log\bigl(e^{-t}+1\bigr) - \tfrac{1}{2}e^{-t}\log\bigl(1+e^t\bigr)\,.$$

Remark. The two general solutions found above differ slightly. However, they are equivalent in the sense that both contain all solutions of the nonhomogeneous equation. The point is that the c_1 and c_2 that appear in each of them are not the same c_1 and c_2. Can you find a relationship between them?

(8) What answer will be produced by the following MATLAB commands?

```
>> ode1 = 'D2y + 2*Dy + 5*y = 16*exp(t)';
>> dsolve(ode1, 't')
ans =
```

Solution. The commands ask MATLAB for a real general solution of the equation

$$\mathrm{D}^2 y + 2\mathrm{D}y + 5y = 16e^t\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

MATLAB will produce something equivalent to the answer

```
2*exp(t) + C1*exp(-t)*sin(2*t) + C2*exp(-t)*cos(2*t)
```

This can be seen as follows. This is a *nonhomogeneous* linear equation for $y(t)$ with *constant* coefficients. The characteristic polynomial is

$$p(z) = z^2 + 2z + 5 = (z+1)^2 + 4 = (z+1)^2 + 2^2 \,.$$

It has the conjugate pair of simple roots $-1 \pm i2$. A real general solution of the associated homogeneous problem is

$$y_H(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) \,.$$

The forcing $16e^t$ has degree $d = 0$, characteristic $\mu + i\nu = 1$, and multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients.

Below we show that each of these methods gives the particular solution $y_P(t) = 2e^t$. Therefore a real general solution is

$$y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + 2e^t \,.$$

Up to notational differnces, this is the answer that MATLAB produces. Your answer does not have to be given in MATLAB format!

Key Indentity Evaluations. Because $d = m = 0$, we only need to evaluate the Key Identity at $z = \mu + i\nu = 1$, to find

$$\mathrm{L}(e^t) = p(1)e^t = (1^2 + 2\cdot 1 + 5)e^t = 8e^t \,.$$

Multiply this by 2 to obtain $\mathrm{L}(2e^t) = 16e^t$. Hence, $y_P(t) = 2e^t$.

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big) \,,$$

this formula gives the particular solution

$$y_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right) \,.$$

For this problem $f(t) = 16e^t$ and $p(z) = z^2 + 2z + 5$, so that $\mu = 1$, $\nu = 0$, $\alpha = 16$, $\beta = 0$, and $m = 0$, whereby

$$y_P(t) = e^t\,\frac{16}{p(1)} = \frac{16}{1^2 + 2\cdot 1 + 5}\,e^t = \frac{16}{8}\,e^t = 2e^t \,.$$

Undetermined Coefficients. Because $m + d = 0$, $m = 0$, and $\mu + i\nu = 1$, there is a particular solution in the form

$$y_P(t) = Ae^t \,.$$

Because

$$y_P'(t) = Ae^t \,, \qquad y_P''(t) = Ae^t \,,$$

we see that

$$\mathrm{L}y_P(t) = y_P''(t) + 2y_P'(t) + 5y_P(t) = [Ae^t] + 2[Ae^t] + 5[Ae^t] = 8Ae^t \,.$$

Setting $\mathrm{L}y_P(t) = 8Ae^t = 16e^t$, we see that $A = 2$. Hence, $y_P(t) = 2e^t$.

(9) Let $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Consider the equation

$$\mathrm{L}r = \mathrm{D}^2 r - 6\mathrm{D}r + 25r = e^{t^2} .$$

(a) Compute the Green function $g(t)$ associated with L.

Solution. The Green function $g(t)$ satisfies

$$\mathrm{D}^2 g - 6\mathrm{D}g + 25g = 0 , \qquad g(0) = 0 , \quad g'(0) = 1 .$$

The characteristic polynomial of L is $p(z) = z^2 - 6z + 25 = (z-3)^2 + 4^2$, which has roots $3 \pm i4$. Set $g(t) = c_1 e^{3t}\cos(4t) + c_2 e^{3t}\sin(4t)$. The first initial condition implies $g(0) = c_1 = 0$, whereby $g(t) = c_2 e^{3t}\sin(4t)$. Because $g'(t) = 3c_2 e^{3t}\sin(4t) + 4c_2 e^{3t}\cos(4t)$, the second initial condition implies $g'(0) = 4c_2 = 1$, whereby $c_2 = \frac{1}{4}$. Therefore the Green function associated with L is given by

$$g(t) = \tfrac{1}{4} e^{3t} \sin(4t) .$$

(b) Use the Green function to express a particular solution $R_P(t)$ in terms of definite integrals.

Solution. A particular solution $R_P(t)$ is given by

$$R_P(t) = \int_0^t g(t-s) e^{s^2} \,\mathrm{d}s = \tfrac{1}{4} \int_0^t e^{3(t-s)} \sin\bigl(4(t-s)\bigr) e^{s^2} \,\mathrm{d}s .$$

Because $\sin\bigl(4(t-s)\bigr) = \sin(4t)\cos(4s) - \cos(4t)\sin(4s)$, this particular solution is given in terms of definite integrals as

$$R_P(t) = \tfrac{1}{4} e^{3t} \sin(4t) \int_0^t e^{-3s} \cos(4s) e^{s^2} \,\mathrm{d}s - \tfrac{1}{4} e^{3t} \cos(4t) \int_0^t e^{-3s} \sin(4s) e^{s^2} \,\mathrm{d}s .$$

Remark. The above definite integrals cannot be evaluated analytically.

(10) The functions t and t^2 are solutions of the homogeneous equation

$$t^2 \frac{\mathrm{d}^2 p}{\mathrm{d}t^2} - 2t \frac{\mathrm{d}p}{\mathrm{d}t} + 2p = 0 \qquad \text{over } t > 0 .$$

(You do not have to check that this is true!)

(a) Compute their Wronskian.

Solution. The Wronskian is

$$\mathrm{Wr}[t, t^2](t) = \det \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = t \cdot (2t) - 1 \cdot t^2 = 2t^2 - t^2 = t^2 .$$

(b) Solve the initial-value problem

$$t^2 \frac{\mathrm{d}^2 q}{\mathrm{d}t^2} - 2t \frac{\mathrm{d}q}{\mathrm{d}t} + 2q = t^3 e^t , \qquad q(1) = q'(1) = 0 , \qquad \text{over } t > 0 .$$

Try to evaluate all definite integrals explicitly.

Solution. Because this problem has *variable* coefficients, we must use either the general Green function method or the variation of parameters method to solve it. To apply either method we must first bring the equation into its normal form

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} - \frac{2}{t}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{2}{t^2}\,q = te^t \qquad \text{over } t > 0\,.$$

Because $\mathrm{Wr}[t, t^2](t) = t^2 \neq 0$ over $t > 0$, we know that t and t^2 constitute a fundamental set of solutions to the associated homogeneous equation.

General Green Function. The Green function $G(t, s)$ is given by

$$G(t,s) = \frac{\det\begin{pmatrix} s & s^2 \\ t & t^2 \end{pmatrix}}{\det\begin{pmatrix} s & s^2 \\ 1 & 2s \end{pmatrix}} = \frac{st^2 - ts^2}{2s^2 - s^2} = \frac{st(t-s)}{s^2} = \frac{t}{s}\,(t-s)\,.$$

The Green function formula then yields the solution

$$\begin{aligned} q(t) &= \int_1^t G(t,s)\,f(s)\,\mathrm{d}s = \int_1^t \frac{t}{s}\,(t-s)\,se^s\,\mathrm{d}s = t\int_1^t (t-s)e^s\,\mathrm{d}s \\ &= t^2\int_1^t e^s\,\mathrm{d}s - t\int_1^t se^s\,\mathrm{d}s = t^2(e^t - e) - t(t-1)e^t = -et^2 + te^t\,. \end{aligned}$$

Variation of Parameters. A real general solution of the associated homogeneous problem is

$$q_H(t) = c_1 t + c_2 t^2\,.$$

Seek a solution in the form

$$q = u_1(t)t + u_2(t)t^2\,,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy

$$\begin{aligned} u_1'(t)t + u_2'(t)t^2 &= 0\,, \\ u_1'(t)1 + u_2'(t)2t &= t\,e^t\,. \end{aligned}$$

Solve this system to obtain

$$u_1'(t) = -t\,e^t\,, \qquad u_2'(t) = e^t\,.$$

Integrate these equations to find

$$u_1(t) = c_1 + (1-t)e^t\,, \qquad u_2(t) = c_2 + e^t\,.$$

Therefore a real general solution is

$$q(t) = c_1 t + c_2 t^2 + (1-t)e^t t + e^t t^2 = c_1 t + c_2 t^2 + t\,e^t\,.$$

Because

$$q'(t) = c_1 + 2c_2 t + (t+1)e^t\,,$$

when the initial conditions are imposed we find that

$$q(1) = c_1 + c_2 + e = 0\,, \qquad q'(1) = c_1 + 2c_2 + 2e = 0\,.$$

These are solved to obtain $c_1 = 0$ and $c_2 = -e$. Therefore the solution of the initial-value problem is

$$q(t) = -et^2 + t\,e^t\,.$$

(11) The vertical displacement of a mass on a spring is given by

$$h(t) = 4e^{-t}\cos(7t) - 3e^{-t}\sin(7t)\,,$$

where positive displacements are upward.

(a) Express $h(t)$ in the form $h(t) = Ae^{-t}\cos(\nu t - \delta)$ with $A > 0$ and $0 \le \delta < 2\pi$, identifying the damped period (quasiperiod) and phase of the oscillation. (The phase may be expressed in terms of an inverse trig function.)

(b) Express $h(t)$ in the phasor form $h(t) = \mathrm{Re}\big(\overline{\gamma}\,e^{\zeta t}\big)$ where γ and ζ are complex numbers.

(c) Sketch the solution over $0 \le t \le 2$.

Solution (a). By comparing

$$Ae^{-t}\cos(\nu t - \delta) = Ae^{-t}\cos(\delta)\cos(\nu t) + Ae^{-t}\sin(\delta)\sin(\nu t)\,,$$

with $h(t) = 4e^{-t}\cos(7t) - 3e^{-t}\sin(7t)$, we see that $\nu = 7$ and that

$$A\cos(\delta) = 4\,, \qquad A\sin(\delta) = -3\,.$$

This shows that (A, δ) are the polar coordinates of the point in the plane whose Cartesian coordinates are $(4, -3)$. Clearly A is given by

$$A = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5\,.$$

Because $(4, -3)$ lies in the fourth quadrant, the phase δ satisfies $\frac{3\pi}{2} < \delta < 2\pi$. There are several ways to express δ. A picture shows that if we use 2π as a reference then

$$\sin(2\pi - \delta) = \tfrac{3}{5}\,, \qquad \tan(2\pi - \delta) = \tfrac{3}{4}\,, \qquad \cos(2\pi - \delta) = \tfrac{4}{5}\,,$$

and we can express the phase by any one of the formulas

$$\delta = 2\pi - \sin^{-1}\big(\tfrac{3}{5}\big)\,, \qquad \delta = 2\pi - \tan^{-1}\big(\tfrac{3}{4}\big)\,, \qquad \delta = 2\pi - \cos^{-1}\big(\tfrac{4}{5}\big)\,.$$

Alternatively, if we use $\frac{3\pi}{2}$ as a reference then

$$\sin(\delta - \tfrac{3\pi}{2}) = \tfrac{4}{5}\,, \qquad \tan(\delta - \tfrac{3\pi}{2}) = \tfrac{4}{3}\,, \qquad \cos(\delta - \tfrac{3\pi}{2}) = \tfrac{3}{5}\,,$$

and we can express the phase by any one of the formulas

$$\delta = \tfrac{3\pi}{2} + \sin^{-1}\big(\tfrac{4}{5}\big)\,, \qquad \delta = \tfrac{3\pi}{2} + \tan^{-1}\big(\tfrac{4}{3}\big)\,, \qquad \delta = \tfrac{3\pi}{2} + \cos^{-1}\big(\tfrac{3}{5}\big)\,.$$

Finally, because the damped frequency (quasifrequency) is $\nu = 7$, the damped period (quasiperiod) T is given by

$$T = \frac{2\pi}{\nu} = \frac{2\pi}{7}\,.$$

Solution (b). Let $\gamma = \alpha + i\beta$ and $\zeta = \mu + i\nu$ where α, β, μ and ν are real numbers. Then the phasor form expands as

$$\begin{aligned}\mathrm{Re}\big(\overline{\gamma}\, e^{\zeta t}\big) &= \mathrm{Re}\big(\overline{(\alpha + i\beta)}\, e^{(\mu+i\nu)t}\big) = \mathrm{Re}\big((\alpha - i\beta)\, e^{\mu t} e^{i\nu t}\big) \\ &= e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)\,(\cos(\nu t) + i\sin(\nu t))\big) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)\,.\end{aligned}$$

By comparing this with

$$h(t) = 4e^{-t}\cos(7t) - 3e^{-t}\sin(7t)\,,$$

we see that $\alpha = 4$, $\beta = -3$, $\mu = -1$, and $\nu = 7$. Therefore

$$h(t) = \mathrm{Re}\big(\overline{\gamma}\, e^{\zeta t}\big) \quad \text{where} \quad \gamma = 4 - i3 \quad \text{and} \quad \zeta = -1 + i7\,.$$

Solution (c). This will be shown during the review session if someone asks for it.

(12) When a 4 gram mass is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) At $t = 0$ the mass is displaced 3 cm above its rest position and released with no initial velocity. A dashpot imparts a damping force of 2 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. There are no other forces. (Assume that the spring force is proportional to displacement and that the damping force is proportional to velocity.)

(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

Solution. Let $h(t)$ be the displacement of the mass from its rest position at time t in centimeters, with upward displacements being positive. The governing initial-value problem then has the form

$$m\ddot{h} + c\dot{h} + kh = 0\,, \qquad h(0) = 3\,, \quad \dot{h}(0) = 0\,,$$

where m is the mass, c is the damping coefficient, and k is the spring constant. The problem says that $m = 4$ grams. The spring constant is obtained by balancing the weight of the mass ($mg = 4 \cdot 980$ dynes) with the force applied by the spring when it is stetched 9.8 cm. This gives $k\,9.8 = 4 \cdot 980$, or

$$k = \frac{4 \cdot 980}{9.8} = 400 \quad \text{dynes/cm}\,.$$

The damping coefficient is obtained by balancing the force of 2 dynes with the damping force imparted by the dashpot when the speed of the mass is 4 cm/sec. This gives $c\,4 = 2$, or

$$c = \tfrac{2}{4} = \tfrac{1}{2} \quad \text{dynes sec/cm}\,.$$

Therefore the governing initial-value problem is

$$4\ddot{h} + \tfrac{1}{2}\,\dot{h} + 400h = 0\,, \qquad h(0) = 3\,, \quad \dot{h}(0) = 0\,.$$

Remark. With the equation in normal form the answer is

$$\ddot{h} + \tfrac{1}{8}\,\dot{h} + 100h = 0\,, \qquad h(0) = 3\,, \quad \dot{h}(0) = 0\,.$$

Remark. If we had chosen downward displacements to be positive then the governing initial-value problem would be the same except for the first initial condition, which would then be $h(0) = -3$.

(b) Find the natural frequency of the spring.

Solution. The natural frequency of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \cdot 980}{4 \cdot 9.8}} = \sqrt{100} = 10 \quad \text{rad/sec}\,.$$

(c) Show that the system is under damped.

Solution. The characteristic polynomial is

$$p(z) = z^2 + \frac{1}{8}z + 100 = \left(z + \frac{1}{16}\right)^2 + 100 - \frac{1}{16^2}\,,$$

which has a conjugate pair of roots. Therefore the system is under damped.

(d) Find the damped frequency (quasifrequency) of the system.

Solution. The roots are $-\frac{1}{16} \pm i\nu$, where the damped frequency (quasifrequency) ν is given by

$$\nu = \sqrt{100 - \frac{1}{16^2}} \quad \text{rad/sec}\,.$$