Final Exam, Math 246/246H
Saturday, 12 May 2018

Closed book. No electronics. Answer only one question on each answer sheet. Write your name and which question is being answered on each answer sheet. Sign the Honor Pledge on the first answer sheet only. Indicate your answer to each part of each question clearly. Indicate on the front of an answer sheet if your work continues on the back. Cross out work that you do not want considered. Give reasoning that justifies your answers.

(1) [22] Find an explicit solution to each of the following initial-value problems. Identify their intervals of definition.
  (a) [11] $v' = e^{-5t}v^6\,, \qquad v(0) = -1\,.$
  (b) [11] $h' + 3t^2 h = e^{-t^3}\cos(t)\,, \qquad h(0) = 2\,.$

(2) [12] Consider the following Matlab commands.

```
[t,y] = ode45(@(t,y) y.*(y−3).*(y−6), [0,5], −3.0:4.0:9.0);
plot(t,y)
```

The following questions need not be answered in Matlab format!
  (a) [2] What is the differential equation being solved numerically?
  (b) [4] Give the initial condition for each solution being approximated.
  (c) [2] Over what time interval are the solutions being approximated?
  (d) [4] Sketch each of these solutions over this time interval on a single graph. Label the initial value of each solution clearly.

(3) [22] Give an explicit real-valued general solution to each of the following equations.
  (a) [11] $x'''' + 8x'' + 16x = 10e^{-t}$
  (b) [11] $u'' + 3u' - 4u = 20e^{t}$

(4) [12] Given that t^2 and t^3 solve the associated homogeneous differential equation, solve the initial-value problem

$$t^2 w'' - 4tw' + 6w = \frac{t^4}{1+t}\,, \qquad w(1) = 0\,, \quad w'(1) = 0\,.$$

Evaluate any definite integrals that arise.

(5) [22] Let $y(t)$ be the solution of the initial-value problem

$$y'' + 4y' + 29y = f(t)\,, \qquad y(0) = 3\,, \quad y'(0) = -4\,,$$

where $f(t) = t^2 + u(t-2)\big(4 - t^2\big)$. Here u is the unit step function.
  (a) [11] Find the Laplace transform $F(s)$ of the forcing $f(t)$.
  (b) [11] Find the Laplace transform $Y(s)$ of the solution $y(t)$.
  (DO NOT take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

You may refer to the table.

More Problems on the Other Side!

(6) [11] Find the function $x(t)$ whose Laplace transform is given by $X(s) = \dfrac{e^{-3s}(s+16)}{s^2+4s+53}$. You may refer to the table.

(7) [11] Two interconnected tanks, each with a capacity of 75 liters, contain brine (salt water). Initially the first tank contains 50 liters and the second tank contains 12 liters. Brine with a salt concentration of 3 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 4 liters per hour, from the first into a drain at 5 liter per hour, and from the second into a drain at 2 liters per hour. Initially there are 17 grams of salt in the first tank and 23 grams in the second tank.

(a) [8] Give an initial-value problem that governs the grams of salt in each tank as a function of time.

(b) [3] Give the interval of definition for the solution of this initial-value problem.

(8) [22] A real 2×2 matrix $\mathbf{B}$ has the eigenpairs

$$\left(1\,,\begin{pmatrix}3\\1\end{pmatrix}\right) \quad \text{and} \quad \left(2\,,\begin{pmatrix}-1\\3\end{pmatrix}\right).$$

(a) [4] Give a general solution to the linear planar system $\mathbf{x}' = \mathbf{B}\mathbf{x}$.

(b) [4] Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ that diagonalize $\mathbf{B}$.

(c) [8] Compute $e^{t\mathbf{B}}$.

(d) [6] Sketch a phase-plane portrait for this system and identify its type. Classify the stability of the origin. Carefully mark all sketched orbits with arrows!

(9) [22] Solve the initial-value problem $\mathbf{x}' = \mathbf{C}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^{\mathrm{I}}$ for the following $\mathbf{C}$ and $\mathbf{x}^{\mathrm{I}}$.

(a) [11] $\mathbf{C} = \begin{pmatrix}1 & 2\\-5 & 3\end{pmatrix}$, $\qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix}2\\1\end{pmatrix}$.

(b) [11] $\mathbf{C} = \begin{pmatrix}2 & -3\\3 & -4\end{pmatrix}$, $\qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix}3\\0\end{pmatrix}$.

(10) [22] Consider the system

$$x' = -y - x^2 + 9\,, \qquad y' = 2xy\,.$$

(a) [11] Find a nonconstant function $H(x,y)$ such that every orbit of the system satisfies $H(x,y) = c$ for some constant c.

(b) [11] The stationary points are $(-3,0)$, $(3,0)$, and $(0,9)$. In the phase-plane sketch these stationary points plus the level set $H(x,y) = c$ for each value of c that corresponds to a stationary point that is a saddle. (No arrows are required!)

(11) [22] Consider the system

$$\dot{u} = -2u + v\,, \qquad \dot{v} = -3u - v - 5u^2\,.$$

(a) [4] This system has two stationary points. Find them.

(b) [6] Find the Jacobian matrix at each stationary point.

(c) [6] Classify the type and stability of each stationary point.

(d) [6] Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. (Carefully mark all sketched orbits with arrows!)