Third In-Class Exam
Math 246, Professor David Levermore
Tuesday, 24 April 2018

Your Name: ______________________________

UMD SID: ______________________________

Discussion Instructor (circle one): Kilian Cooley Corry Bedwell Thien Ngo
Discussion Time (circle one): 8:00 9:00 10:00 11:00

No books, notes, calculators, or any electronic devices. If you need more space to answer a problem then use the back of one of these pages. Clearly indicate where your answer to each part of every problem is located. Any work that you do not want to be considered should be crossed out. **Your reasoning must be given for full credit.** Good luck!

University Honor Pledge: *I pledge on my honor that I have not given or received any unauthorized assistance on this examination.* ______________________________
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Signature: ______________________________

Problem 1:	____/6	Problem 2:	____/10
Problem 3:	____/6	Problem 4:	____/10
Problem 5:	____/6	Problem 6:	____/8
Problem 7:	____/8	Problem 8:	____/8
Problem 9:	____/10	Problem 10:	____/8
Problem 11:	____/12	Problem 12:	____/8
		Total Score:	____/100
		Grade:	____

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(1) [6] Recast the ordinary differential equation $y'''' = e^y y''' + (y'')^2 + \cos(t^3 + y')$ as a first-order system of ordinary differential equations.

(2) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 4 \\ 3t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1+t^4 \end{pmatrix}$.

(a) [2] Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system found in part (b).

(d) [3] Compute the Green matrix associated with the system found in part (b).

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(3) [6] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 26 liters and the second contains 19 liters. Brine with a salt concentration of 5 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 8 liters per hour, from the second into the first at 7 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 2 liters per hour. At $t = 0$ there are 17 grams of salt in the first tank and 31 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(4) [10] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

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(5) [6] Given that 2 is an eigenvalue of the matrix

$$\mathbf{B} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 2 & -3 \\ 0 & 2 & 10 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{B}$ associated with 2.

(6) [8] A 4×4 matrix $\mathbf{C}$ has the eigenpairs

$$\left(5\,, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right), \qquad \left(2\,, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}\right), \qquad \left(-1\,, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right), \qquad \left(-4\,, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\right).$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{C}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{C}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{C}\mathbf{x}$.

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(7) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -4 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(8) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

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(9) [10] Find the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{L} = \mathrm{D}^3 + 4\mathrm{D}$.

(10) [8] Compute the Laplace transform of $f(t) = u(t-5)\, e^{-3t}$ from its definition. (Here u is the unit step function.)

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(11) [12] Consider the following (old style) MATLAB commands.

```
>> syms t s Y; f = ['t^2 + heaviside(t - 2)*(4 - t^2) - heaviside(t - 6)*4'];
>> diffeqn = sym('D(D(y))(t) + 6*D(y)(t) + 34*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s)', 'y(0)', 'D(y)(0)'}, {Y, 4, -2});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) [4] Give the initial-value problem for $y(t)$ that is being solved.
(b) [8] Find the Laplace transform $Y(s)$ of the solution $y(t)$. (DO NOT take the inverse Laplace transform of $Y(s)$ to find $y(t)$, just solve for $Y(s)$!)

You may refer to the table on the last page.

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[X(s)](t)$ of the function

$$X(s) = e^{-4s}\,\frac{3s+11}{s^2+6s+13}\,.$$

You may refer to the table on the last page.

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Workspace (Give the number of the problem being worked!)

Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s) \text{ and } u \text{ is the unit step function}\,.$$

$$\mathcal{L}[\delta(t-c)h(t)](s) = e^{-cs}h(c) \qquad \text{where } \delta \text{ is the unit impluse}\,.$$