Final Exam: MATH 410
Thursday, 14 December 2017
Professor David Levermore

1. [10] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Give negations of each of the following assertions.
   (a) For every $\epsilon > 0$ there exists an $n_\epsilon \in \mathbb{N}$ such that
   $$m, n > n_\epsilon \implies |x_m - x_n| < \epsilon\,.$$
   (b) $\lim_{n\to\infty} x_n = \infty\,.$

2. [15] Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be bounded, positive sequences in $\mathbb{R}$.
   (a) [10] Prove that
   $$\limsup_{k\to\infty}\,(a_k b_k) \le \Big(\limsup_{k\to\infty}\, a_k\Big)\Big(\limsup_{k\to\infty}\, b_k\Big)\,.$$
   (b) [5] Give an example for which equality does not hold above.

3. [15] Let $f : (a,b) \to \mathbb{R}$ be differentiable at a point $c \in (a,b)$ with $f'(c) < 0$. Show that there exists a $\delta > 0$ such that
   $$x \in (c-\delta, c) \subset (a,b) \implies f(x) > f(c)\,,$$
   $$x \in (c, c+\delta) \subset (a,b) \implies f(c) > f(x)\,,$$

4. [20] Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be Riemann integrable over $[a,b]$. Prove that $f+g$ is Riemann integrable over $[a,b]$.

5. [25] Consider a function g defined by
   $$g(x) = \sum_{k=0}^{\infty} \frac{1}{3^k}\sin(2^k x)\,,$$
   for every $x \in \mathbb{R}$ for which the above series converges.
   (a) [10] Show that g is defined for every $x \in \mathbb{R}$.
   (b) [15] Show that g is continuously differentiable over $\mathbb{R}$ and that
   $$g'(x) = \sum_{k=0}^{\infty} \frac{2^k}{3^k}\cos(2^k x)\,.$$

6. [25] For every $n \in \mathbb{Z}_+$ define $h_n(x) = nx(1+nx)^{-2}$ for every $x \in [0,\infty)$.
   (a) [5] Prove that $h_n \to 0$ pointwise over $[0,\infty)$.
   (b) [10] Prove that this limit is not uniform over $[0,\infty)$.
   (c) [10] Prove that this limit is uniform over $[\delta,\infty)$ for every $\delta > 0$.

There are more problems on the other side.

7. [20] Determine the set of $a \in \mathbb{R}$ for which the following formal infinite series converge. Give your reasoning.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{n3^n}$

   (b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k^2+1}{k^4+1}\right)^a$

8. [20] Let $\alpha \in (0,1]$ and $K \in \mathbb{R}_+$ such that the function $f : [a,b] \to \mathbb{R}$ satisfy the Hölder bound

$$|f(x) - f(y)| < K\,|x-y|^\alpha \qquad \text{for every } x, y \in [a,b]\,.$$

   (a) Show that f is uniformly continuous over $[a,b]$.

   (b) Show that for every partition P of $[a,b]$ one has

$$0 \leq U(f,P) - L(f,P) < |P|^\alpha K\,(b-a)\,.$$

9. [25] Given the fact that for every $x > -1$ and every $n \in \mathbb{Z}_+$ we have

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \log(1+x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}\,,$$

   prove that

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\, x^k \quad \text{for every } x \in (-1,1]\,,$$

   and that the series diverges for every real $x \notin (-1,1]$.

10. [25] For every $n \in \mathbb{Z}_+$ define $f_n(x) = n(1+nx)^{-2}$ for every $x \in [0,\infty)$.

   (a) [10] Prove for every $\delta > 0$ that

$$\lim_{n\to\infty} f_n = 0 \qquad \text{uniformly over } [\delta,\infty)\,.$$

   (b) [5] Prove for every $\delta > 0$ that

$$\lim_{n\to\infty} \int_0^\delta f_n = 1\,.$$

   (c) [10] Let $g : [0,1] \to \mathbb{R}$ be continuous. Show that

$$\lim_{n\to\infty} \int_0^1 f_n g = g(0)\,.$$