Quiz 1 Solutions, Math 246, Professor David Levermore
Thursday, 31 August 2017

(1) [4] For each of the following ordinary differential equations, determine its order and whether it is linear or nonlinear. If it is nonlinear, circle a term that makes it so.

(a) $v'''' + \sin(v)v' = 5e^t$ **Solution.** fourth order, nonlinear, $\sin(v)v'$.

(b) $k''' + \cos(t)k'' = 3k + \sin(3t)$ **Solution.** third order, linear.

(2) [4] Solve the initial-value problem

$$z\frac{\mathrm{d}u}{\mathrm{d}z} + 4u = 12z^2\,, \qquad u(-1) = 5\,.$$

Solution. This equation is linear. Its normal form is

$$\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{4}{z}\,u = 12z\,.$$

An integrating factor is $e^{A(z)}$ where $A'(z) = 4/z$. Setting $A(z) = 4\log(|z|)$, we obtain $e^{A(z)} = e^{4\log(|z|)} = z^4$. Hence, the problem has the intgrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}z}\big(z^4 u\big) = z^4 \cdot (12z) = 12z^5\,.$$

Integrating both sides yields

$$z^4 u = 2z^6 + c\,.$$

Imposing the initial condition gives

$$(-1)^4 \cdot 5 = 2 \cdot (-1)^6 + c\,,$$

whereby $c = 3$. Therefore the solution is

$$u = \frac{2z^6 + 3}{z^4}\,.$$

Remark. The interval of definition for this solution is $(-\infty, 0)$. Do you see why?

(3) [2] What is the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\sin(x)}{x^2 - 9}\,w = \frac{e^x}{x^2 - 4}\,, \qquad w(-5) = 7\,.$$

(You do not have to solve the differential equation to answer this question!)

Solution. This is a nonhomogeneous linear equation that is already in normal form. The coefficient $\sin(x)/(x^2 - 9)$ is undefined at $x = \pm 3$ and is continuous elsewhere. The forcing $e^x/(x^2 - 4)$ is undefined at $x = \pm 2$ and is continuous elsewhere. The initial time is $x = -5$. This can be pictured on the x-axis as follows.

$$\begin{array}{l} - - \bullet - - + - - \circ - - \circ - - + - - + - - + - - \circ - - \circ - - + - - + - \rightarrow\ x \\ \ \ -5 \quad\ -4 \quad\ -3 \quad\ -2 \quad\ -1 \quad\ \ 0 \quad\ \ \ 1 \quad\ \ \ 2 \quad\ \ \ 3 \quad\ \ \ 4 \quad\ \ \ 5 \end{array}$$

Therefore the interval of definition for the solution is $(-\infty, -3)$ because:

- the initial time $x = -5$ is in $(-\infty, -3)$,
- the coefficient is undefined at $x = -3$,
- the coefficient and forcing are both continuous over $(-\infty, -3)$.

Question. How does the answer change if the initial condition is $w(1) = -4$?