Quiz 9 Solutions, Math 246, Professor David Levermore
Thursday, 20 April 2017

(1) [3] Give the interval of definition for solutions of the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \tan(t)\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(2) \\ y(2) \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

Solution. This is a homogeneous linear system in normal form. Its coefficient matrix is

$$\tan(t)\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 4\tan(t) & -\tan(t) \\ -2\tan(t) & 3\tan(t) \end{pmatrix}.$$

Its entries are undefined at $t = \frac{\pi}{2} + n\pi$ for every integer n and are continuous elsewhere. The initial time is $t = 2$. Because $\frac{\pi}{2} < 2 < \frac{3\pi}{2}$, the interval of definition is $(\frac{\pi}{2}, \frac{3\pi}{2})$.

(2) [7] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^2 \\ -1 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

(a) [2] Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.
(b) [4] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.
(c) [1] Give a general solution the system that you found in part (b).

Solution (a). The Wronskian is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^2 & e^t \\ -1 & e^t \end{pmatrix} = t^2 \cdot e^t - (-1)\cdot e^t = (t^2+1)e^t.$$

Solution (b). Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^2 & e^t \\ -1 & e^t \end{pmatrix}$. Because $\mathbf{\Psi}'(t) = \mathbf{A}(t)\mathbf{\Psi}(t)$, we have

$$\begin{aligned}
\mathbf{A}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 2t & e^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} t^2 & e^t \\ -1 & e^t \end{pmatrix}^{-1} \\
&= \frac{1}{(1+t^2)e^t}\begin{pmatrix} 2t & e^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} e^t & -e^t \\ 1 & t^2 \end{pmatrix} \\
&= \frac{1}{(1+t^2)e^t}\begin{pmatrix} 2te^t + e^t & -2te^t + t^2e^t \\ e^t & t^2e^t \end{pmatrix} \\
&= \frac{1}{1+t^2}\begin{pmatrix} 2t+1 & t^2 - 2t \\ 1 & t^2 \end{pmatrix}.
\end{aligned}$$

Solution (c). A general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} t^2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Remark. The natural fundamental matrix $\Phi(t)$ associated with an initial time 0 is

$$\Phi(t) = \Psi(t)\Psi(0)^{-1} = \begin{pmatrix} t^2 & e^t \\ -1 & e^t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} t^2 & e^t \\ -1 & e^t \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \cdots.$$