Quiz 7 Solutions, Math 246, Professor David Levermore
Thursday, 6 April 2017

(1) [4] A sping-mass system is governed by the initial-value problem

$$h'' + 2\mu h' + 16h = 0\,, \qquad h(0) = 1\,, \quad h'(0) = 0\,.$$

(a) What is the natural frequency and period of the spring?
(b) For what value of μ is the system critically damped?

Solution (a). Because the equation is in normal form, the natural frequency ω_o is

$$\omega_o = \sqrt{16} = 4\,.$$

The natural period is thereby $T_o = 2\pi/\omega_o = \frac{2\pi}{4} = \frac{\pi}{2}$.

Solution (b). The characteristic polynomial is

$$p(z) = z^2 + 2\mu z + 16 = (z+\mu)^2 + 16 - \mu^2\,.$$

The system will be critically damped when this polynomial has a double real root. This happens when $16 - \mu^2 = 0$, which is when

$$\mu = \sqrt{16} = 4\,.$$

(2) [1] Give the exponential order as $t \to \infty$ of the function $t^4 e^{-2t}\cos(3t)$.

Solution. The exponential orders as $t \to \infty$ of t^4, e^{-2t}, and $\cos(3t)$ are 0, -2, and 0 repsectively. Because the exponential order of a product is the sum of the exponential orders, we see that the exponential order as $t \to \infty$ of $t^4 e^{-2t}\cos(3t)$ is -2.

Alternative Solution. The exponential order as $t \to \infty$ of $t^4 e^{-2t}\cos(3t)$ is -2 because for every $a > -2$ we have

$$\lim_{t\to\infty} e^{-at} t^4 e^{-2t}\cos(3t) = \lim_{t\to\infty} t^4 e^{-(a+2)t}\cos(3t) = 0\,.$$

(3) [5] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for the function $f(t) = e^{3t}u(t-2)$, where u is the unit step function.

Solution. By the definitions of the Laplace transform and the unit step function

$$\begin{aligned}
\mathcal{L}[f](s) &= \lim_{T\to\infty}\int_0^T e^{-st} f(t)\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T e^{-st} e^{3t} u(t-2)\,\mathrm{d}t \\
&= \lim_{T\to\infty}\int_2^T e^{-st} e^{3t}\,\mathrm{d}t = \lim_{T\to\infty}\int_2^T e^{-(s-3)t}\,\mathrm{d}t\,.
\end{aligned}$$

For $s \leq 3$ the above limit diverges because $e^{-(s-2)t} \geq 1$. For $s > 3$ and $T > 2$

$$\int_2^T e^{-(s-3)t}\,\mathrm{d}t = -\frac{e^{-(s-3)t}}{s-3}\bigg|_2^T = \frac{e^{-(s-3)2}}{s-3} - \frac{e^{-(s-3)T}}{s-3}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[\frac{e^{-(s-3)2}}{s-3} - \frac{e^{-(s-3)T}}{s-3}\right] = \frac{e^{-(s-3)2}}{s-3} \quad \text{for } s > 3\,.$$