Quiz 4 Solutions, Math 246, Professor David Levermore
Thursday, 2 March 2017

(1) [2] What is the interval of definition for the solution to the initial-value problem

$$v''' + \frac{e^t}{1+t}\, v'' - \frac{\sin(2t)}{7-t}\, v = \frac{\cos(3t)}{9+t}\,, \qquad v(-2) = v'(-2) = v''(-2) = -5\,.$$

Solution. This linear equation is in normal form. One of its coefficients is undefined at $t = -1$ and continuous elsewhere while the other is undefined at $t = 7$ and continuous elsewhere. Its forcing is undefined at $t = -9$ and continuous elsewhere. The initial time is $t = -2$. Therefore the interval of definition is $(-9, -1)$.

(2) [3] Compute the Wronskian $W[Z_1, Z_2](t)$ of the functions $Z_1(t) = 1+t$ and $Z_2(t) = e^t$. (Evaluate the determinant and simplify.)

Solution. Because $Z_1'(t) = 1$ and $Z_2'(t) = e^t$, the Wronskian is

$$\begin{aligned} W[Z_1, Z_2](t) &= \det\begin{pmatrix} Z_1(t) & Z_2(t) \\ Z_1'(t) & Z_2'(t) \end{pmatrix} = \det\begin{pmatrix} 1+t & e^t \\ 1 & e^t \end{pmatrix} \\ &= (1+t)e^t - 1e^t = te^t\,. \end{aligned}$$

(3) [1] Suppose that $X_1(t)$, $X_2(t)$, and $X_3(t)$ are solutions of the differential equation

$$x''' + b(t)x' + c(t)x = 0\,,$$

where $b(t)$ and $c(t)$ are continuous over $(-7, 8)$. Suppose that $W[X_1, X_2, X_3](2) = -2$. What is $W[X_1, X_2, X_3](-5)$?

Solution. The equation is in normal form and has coefficients that are continuous over $(-7, 8)$. Because it is third-order while the coefficient of x'' is zero, the Abel Theorem implies that $W[X_1, X_2, X_3](t)$ is constant over $(-7, 8)$. We thereby conclude that $W[X_1, X_2, X_3](-5) = W[X_1, X_2, X_3](2) = -2$.

(4) [4] Given that e^{3t} and e^{-3t} are linearly independent solutions of $u'' - 9u = 0$, find the natural fundamental set of solutions of this equations associated with $t = 0$.

Solution. The general initial-value problem associated with $t = 0$ is

$$u'' - 9u = 0\,, \qquad u(0) = u_0\,, \quad u'(0) = u_1\,.$$

Because e^{3t} and e^{-3t} are linearly independent solutions, we see a general solution is $U(t) = c_1 e^{3t} + c_2 e^{-3t}$. Then $U'(t) = 3c_1 e^{3t} - 3c_2 e^{-3t}$ and the initial conditions yield

$$u_0 = U(0) = c_1 + c_2\,, \qquad u_1 = U'(0) = 3c_1 - 3c_2\,.$$

It follows that $3u_0 + u_1 = 6c_1$ and $3u_0 - u_1 = 6c_2$, whereby

$$c_1 = \tfrac{1}{2}u_0 + \tfrac{1}{6}u_1\,, \qquad c_2 = \tfrac{1}{2}u_0 - \tfrac{1}{6}u_1\,.$$

Hence, the solution of the general initial-value problem is

$$U(t) = \left(\tfrac{1}{2}u_0 + \tfrac{1}{6}u_1\right) e^{3t} + \left(\tfrac{1}{2}u_0 - \tfrac{1}{6}u_1\right) e^{-3t} = \tfrac{1}{2}(e^{3t} + e^{-3t})u_0 + \tfrac{1}{6}(e^{3t} - e^{-3t})u_1\,.$$

Therefore the natural fundamental set of solutions associated with $t = 0$ is

$$N_0(t) = \tfrac{1}{2}(e^{3t} + e^{-3t})\,, \qquad N_1(t) = \tfrac{1}{6}(e^{3t} - e^{-3t})\,.$$