Final Exam Sample Problems, Math 246, Spring 2017

(1) Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = (9 - y^2)y^2$.
 (a) Identify its stationary points and classify their stability.
 (b) Sketch its phase-line portrait in the interval $-5 \le y \le 5$.
 (c) If $y(0) = -1$, how does the solution $y(t)$ behave as $t \to \infty$?

(2) Find an explicit solution to each of the following initial-value problems. Identify their intervals of definition.

 (a) $\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2ty}{1+t^2} = t^2 , \quad y(0) = 1 .$
 (b) $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{e^x y + 2x}{2y + e^x} = 0 , \quad y(0) = 0 .$

(3) Consider the following Matlab function m-file.

```
function [t,y] = solveit(ti, yi, tf, n)
t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = ti; y(1) = yi; h = (tf - ti)/n;
for i = 1:n
t(i + 1) = t(i) + h; y(i + 1) = y(i) + h*((t(i))^4 + (y(i))^2);
end
```

Suppose that the input values are ti = 1, yi = 1, tf = 5, and n = 40.
 (a) What initial-value problem is being approximated numerically?
 (b) What numerical method is being used?
 (c) What is the step size?
 (d) What are the output values of t(2), y(2), t(3), and y(3)?

(4) Consider the following Matlab commands.

```
[t,y] = ode45(@(t,y) y.*(y−1).*(2−y), [0,3], -0.5:0.5:2.5);
plot(t,y)
```

The following questions need not be answered in Matlab format!
 (a) What is the differential equation being solved numerically?
 (b) Give the initial condition for each solution being approximated?
 (c) Over what time interval are the solutions being approximated?
 (d) Sketch each of these solutions over this time interval on a single graph. Label the initial value of each solution clearly.
 (e) What is the limiting behavior of each solution as $t \to \infty$?

(5) Give an explicit real-valued general solution of the following equations.
 (a) $y'' - 2y' + 5y = te^t + \cos(2t)$
 (b) $u'' - 3u' - 10u = te^{-2t}$
 (c) $v'' + 9v = \cos(3t)$

(6) Solve the following initial-value problems.

(a) $w'' + 4w' + 20w = 5e^{2t}, \qquad w(0) = 3\,, \quad w'(0) = -7.$

(b) $y'' - 4y' + 4y = \dfrac{e^{2t}}{3+t}, \qquad y(0) = 0\,, \quad y'(0) = 5.$

Evaluate any definite integrals that arise.

(7) Give an explicit real-valued general solution of the equation

$$h'' + 2h' + 5h = 0\,.$$

Sketch a typical solution for $t \geq 0$. If this equation governs a spring-mass system, is the system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

(8) When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.5 m. (Gravitational acceleration is 9.8 m/sec^2.) At $t = 0$ the mass is set in motion from 0.3 meters below its rest (equilibrium) position with a upward velocity of 2 m/sec. It is acted upon by an external force of $2\cos(5t)$. Neglect drag and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem; just write it down!)

(9) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial-value problem

$$y'' + 4y' + 8y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 4\,.$$

where

$$f(t) = \begin{cases} 4 & \text{for } 0 \leq t < 2\,, \\ t^2 & \text{for } 2 \leq t\,. \end{cases}$$

You may refer to the table of Laplace transforms on the last page. (DO NOT take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

(10) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by

(a) $Y(s) = \dfrac{e^{-3s}4}{s^2 - 6s + 5}\,, \qquad$ (b) $Y(s) = \dfrac{e^{-2s}s}{s^2 + 4s + 8}\,.$

You may refer to the table of Laplace transforms on the last page.

(11) Consider two interconnected tanks filled with brine (salt water). The first tank contains 80 liters and the second contains 30 liters. Brine flows with a concentration of 3 grams of salt per liter flows into the first tank at a rate of 2 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 6 liters per hour, from the second to the first at a rate of 4 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 7 grams of salt in the first tank and 25 grams in the second.

(a) Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(b) Give the interval of definition for the solution of this initial-value problem.

(12) Consider the real vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3 + t^4 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.
(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the linear system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.
(c) Give a general solution to the system you found in part (b).

(13) Give a real, vector-valued general solution of the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for

(a) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

(14) What answer will be produced by the following Matlab command?

```
>> A = [1 4; 3 2]; [vect, val] = eig(sym(A))
```

You do not have to give the answer in Matlab format.

(15) A real 2×2 matrix $\mathbf{B}$ has the eigenpairs

$$\left(2, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right) \quad \text{and} \quad \left(-1, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right).$$

(a) Give a general solution to the linear planar system $\mathbf{x}' = \mathbf{B}\mathbf{x}$.
(b) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ that diagonalize $\mathbf{B}$.
(c) Compute $e^{t\mathbf{B}}$.
(d) Sketch a phase-plane portrait for this system and identify its type. Classify the stability of the origin. Carefully mark all sketched orbits with arrows!

(16) Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^{\mathrm{I}}$ and describe how its solution behaves as $t \to \infty$ for the following $\mathbf{A}$ and $\mathbf{x}^{\mathrm{I}}$.

(a) $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ -5 & -7 \end{pmatrix}$, $\mathbf{x}^{\mathrm{I}} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

(b) $\mathbf{A} = \begin{pmatrix} 8 & -5 \\ 5 & -2 \end{pmatrix}$, $\mathbf{x}^{\mathrm{I}} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

(17) Consider the nonlinear planar system

$$x' = 2xy\,, \qquad y' = 9 - 9x - y^2\,.$$

(a) Find all of its stationary points.
(b) Find a nonconstant function $H(x, y)$ such that every orbit of the system satisfies $H(x, y) = c$ for some constant c.
(c) Classify the type and stability of each stationary point.
(d) Sketch the stationary points plus the level set $H(x, y) = c$ for each value of c that corresponds to a stationary point that is a saddle. (Carefully mark all sketched orbits with arrows!)

(18) Consider the nonlinear planar system

$$u' = -5v\,, \qquad v' = u - 4v - u^2\,.$$

(a) Find all of its stationary points.
(b) Compute the Jacobian matrix at each stationary point.
(c) Classify the type and stability of each stationary point.
(d) Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. (Carefully mark all sketched orbits with arrows!)

(19) Consider the nonlinear planar system

$$p' = p(3 - 3p + 2q)\,, \qquad q' = q(6 - p - q)\,.$$

(a) Find all of its stationary points.
(b) Compute the Jacobian matrix at each stationary point.
(c) Classify the type and stability of each stationary point.
(d) Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. (Carefully mark all sketched orbits with arrows!)
(e) Add the orbits of all semistationary solutions to the phase-plane portrait sketched for part (d). (Carefully mark these sketched orbits with arrows!)
(f) Why do solutions that start in the first quadrant stay in the first quadrant?