Semester Projects

AMSC 460, Section 0201, Fall 2016

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Due 19 December 2016

Consider the function

$$f(x) = \frac{1}{1+x^2} \qquad \text{over the interval } [-4, 4]. \tag{1}$$

Project A: Interpolation

1a. [20pts] Find the following polynomial interpolations of $f(x)$ over the interval $[-4, 4]$:

- the quadratic intrepolation of the points

$$(-4, f(-4))\,, \qquad (0, f(0))\,, \qquad (4, f(4))\,;$$

- the quartic interpolation of the points

$$(-4, f(-4))\,, \qquad (-2, f(-2))\,, \qquad (0, f(0))\,, \qquad (2, f(2))\,, \qquad (4, f(4))\,;$$

- the eighth degree interpolation of the points

$$\begin{gathered}(-4, f(-4))\,, \qquad (-3, f(-3))\,, \qquad (-2, f(-2))\,, \qquad (-1, f(-1))\,, \\ (0, f(0))\,, \qquad (1, f(1))\,, \qquad (2, f(2))\,, \qquad (3, f(3))\,, \qquad (4, f(4))\,.\end{gathered}$$

1b. [10pts] Graph $f(x)$ and these three interpolants over $[-4, 4]$ on a single graph. Discuss which of these might be best.

1c. [10pts] Graph $f'(x)$ and the derivatives of these three interpolants over $[-4, 4]$ on a single graph. Discuss which of these might be best.

1d. [10pts] Graph $f''(x)$ and the second derivatives of these three interpolants over $[-4, 4]$ on a single graph. Discuss which of these might be best.

2. Find the natural cubic spline interpolation $s(x)$ of $f(x)$ over the interval $[-4,4]$ with the knots at the points

$$\begin{gathered}(-4, f(-4))\,, \qquad (-3, f(-3))\,, \qquad (-2, f(-2))\,, \qquad (-1, f(-1))\,, \\ (0, f(0))\,, \qquad (1, f(1))\,, \qquad (2, f(2))\,, \qquad (3, f(3))\,, \qquad (4, f(4))\,.\end{gathered}$$

2a. [20pts] Give the cubics over each of the eight subintervals.

2b. [10pts] Write down the linear alegbraic system that you needed to solve to compute $s''(x)$ at each of the knots.

2c. [5pts] Give $s'(x)$ and $s''(x)$ at each of the knots.

2d. [5pts] Graph $f(x)$ and $s(x)$ over $[-4,4]$ on a single graph. Compare $s(x)$ with the polynomial approximations found in the previous problem.

2e. [5pts] Graph $f'(x)$ and $s'(x)$ over $[-4,4]$ on a single graph. Compare $s'(x)$ with the derivatives of the polynomial approximations found in the previous problem.

2f. [5pts] Graph $f''(x)$ and $s''(x)$ over $[-4,4]$ on a single graph. Compare $s''(x)$ with the second derivatives of the polynomial approximations found in the previous problem.

Project B: Least Squares Approximation

1a. [20pts] Compute the first ten orthogonal monic polynomials (degrees zero through nine) with respect to the scalar product

$$\langle g \mid h \rangle = \int_{-4}^{4} g(x)\, h(x)\, dx\,. \tag{2}$$

1b. [10pts] Normalize these ten polynomials so that they form an orthonomal set.

2a. [20pts] Find the polynomial $p(x)$ of degree at most nine that minimizes

$$\int_{-4}^{4} \big(p(x) - f(x)\big)^2\, dx\,,$$

where $f(x)$ is given by (1).

2b. [5pts] Graph $f(x)$ and $p(x)$ over $[-4, 4]$ on a single graph. Compare $p(x)$ with the approximations found in the previous project.

2c. [5pts] Graph $f'(x)$ and $p'(x)$ over $[-4, 4]$ on a single graph. Compare $p'(x)$ with the derivatives of the approximations found in the previous project.

2d. [5pts] Graph $f''(x)$ and $p''(x)$ over $[-4, 4]$ on a single graph. Compare $p''(x)$ with the second derivatives of the approximations found in the previous project.

3a. [5pts] Consider the functions

$$1\,, \qquad \cos(x\pi/4)\,, \qquad \cos(x\pi/2)\,.$$

Show that they form an orthogonal set with respect to the scalar product (2).

3b. [15pts] Find the linear combination $c(x)$ of these three functions that minimizes

$$\int_{-4}^{4} \big(c(x) - f(x)\big)^2\, dx\,,$$

where $f(x)$ is given by (1).

3c. [5pts] Graph $f(x)$ and $c(x)$ over $[-4, 4]$ on a single graph. Compare $c(x)$ with the approximations found in the previous project.

3d. [5pts] Graph $f'(x)$ and $c'(x)$ over $[-4, 4]$ on a single graph. Compare $c'(x)$ with the derivatives of the approximations found in the previous project.

3e. [5pts] Graph $f''(x)$ and $c''(x)$ over $[-4, 4]$ on a single graph. Compare $c''(x)$ with the second derivatives of the approximations found in the previous project.

Project C: Numerical Quadrature

This project compares various approximations to the definite integral

$$\mathcal{I}[f] = \int_{-4}^{4} f(x)\, dx\,, \tag{3}$$

where $f(x)$ is given by (1).

1. [5pts] Compute the integral (3) exactly.

2. Compute the integral (3) approximately using:

2a. [10pts] the trapezoidal rule with n uniform subintervals;

2b. [10pts] the midpoint rule with n uniform subintervals;

2c. [10pts] the Simpson rule with n uniform subintervals;

In each case pick n large enough so that the error is less than 10^{-6}. Give reasoning for your choice of n.

3a. [30pts] Determine the Gauss quadrature points $\{x_i\}_{i=1}^{n}$ and weights $\{w_i\}_{i=1}^{n}$ for $n = 3$, $n = 5$, and $n = 9$. (If you cannot find exact expressions for these quantities for $n = 9$ then compute them numerically. In any event, make it clear how you found them.)

3b. [10pts] Compute the integral (3) approximately using Gauss quadrature with 3, 5, and 9 points. Compare the results with the exact value.

4. [15pts] Compute the integral (3) approximately using Romberg integration. Stop when the error is within 10^{-8}. Print out the Romberg table of values $R_{m,n}$.

5. [10pts] Compute the integral (3) approximately by exactly integrating the cubic spline interpolation $s(x)$ found in Problem 2 of Project A. (Recall that the Simpson rule integrates cubics exactly, so to integrate $s(x)$ exactly you only have to apply the Simpson rule over $[-4, 4]$ with 8 subintervals.) Compare the result with the exact value.