GAUSS QUADRATURE METHODS

AMSC 460, Section 0201, Fall 2016
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1. Introduction. We consider quadrature formulas in the form

$$\mathcal{Q}[f] = \sum_{j=1}^{m} f(x_j)\, w_j\,, \tag{1.1}$$

that approximate the value of definite integrals in the form

$$\mathcal{I}[f] = \int_a^b f(x)\, w(x)\, dx\,, \tag{1.2}$$

where $(a, b) \subset \mathbb{R}$ and $w(x) > 0$ over (a, b). We will assume that $\mathcal{I}[f]$ is defined whenever f is a polynomial. This means that (a, b) and $w(x)$ must satisfy

$$\int_a^b (1 + x^2)^n\, w(x)\, dx < \infty \qquad \text{for every } n \in \mathbb{N}\,. \tag{1.3}$$

This will be the case whenever the interval (a, b) is bounded and the function $w(x)$ is integrable. It will also be the case when either $a = -\infty$ or $b = \infty$ whenever $w(x)$ decays faster than every negative power of $|x|$ as $x \to -\infty$ or $x \to \infty$ respectively.

We pose the following question.

Given $m \geq 1$, what quadrature points $\{x_j\}_{j=1}^m$ and quadrature weights $\{w_j\}_{j=1}^m$ maximize the precision of quadrature formula (1.1)?

Here we show that there exists a unique set of quadrature points and weights for which the precision of quadrature formula (1.1) is $2m - 1$, and that no higher precision can be attained. This result is due to Gauss for the case $w(x) = 1$, and the associated methods are called *Gaussian* or *Gauss quadrature* methods.

Remark. The quadrature weights $\{w_j\}_{j=1}^m$ associated with Gauss quadrature methods are always positive. This contrasts with closed Newton-Cotes quadrature methods, which have negative weights for $m = 8$ and for every $m \geq 10$.

2. Quadrature Weights. The first fact we establish is that the quadrature weights are determined by the quadrature points whenever the precision of quadrature formula (1.1) is at least $m - 1$.

Fact 1. The precision of quadrature formula (1.1) is at least $m-1$ if and only if the quadrature weights are given by the formula

$$w_j = \mathcal{I}[\ell_j] = \int_a^b \ell_j(x)\, w(x)\, dx \qquad \text{for every } j \in \{1, \cdots, m\}\,, \tag{2.1}$$

where $\ell_j(x)$ is the j^{th} Lagrange interpolating polynomial, which is given by

$$\ell_j(x) = \prod_{\substack{k=1\\k\neq j}}^{m} \frac{x - x_k}{x_j - x_k}\,. \tag{2.2}$$

Remark. When $w(x) = 1$ and the quadrature points x_j are uniformly spaced as

$$x_j = a + \tfrac{j-1}{m-1}(b-a) \qquad \text{for } j = 1, \cdots, m\,,$$

then formula (2.1) yields the m-point closed Newton-Cotes weights. When $w(x) = 1$ and the quadrature points x_j are uniformly spaced as

$$x_j = a + \tfrac{j-\frac{1}{2}}{m}(b-a) \qquad \text{for } j = 1, \cdots, m\,,$$

then formula (2.1) yields the m-point open Newton-Cotes weights.

Proof. ($\Rightarrow$) Let the precision of formula (1.1) be at least $m-1$. Then because the degree of each $\ell_j(x)$ is $m-1$, we have $\mathcal{Q}[\ell_j] = \mathcal{I}[\ell_j]$. Hence, for every $j \in \{1, \cdots, m\}$ we have

$$w_j = \sum_{k=1}^{m} \ell_j(x_k)\, w_k = \mathcal{Q}[\ell_j] = \mathcal{I}[\ell_j] = \int_a^b \ell_j(x)\, w(x)\, dx\,.$$

Therefore (2.1) holds.

($\Leftarrow$) Let (2.1) hold. Let $f(x)$ be any polynomial of degree at most $m-1$. By the Lagrange interpolation formula we have

$$f(x) = \sum_{j=1}^{m} f(x_j)\, \ell_j(x)\,.$$

Hence,

$$\begin{aligned}\mathcal{I}[f] &= \int_a^b \sum_{j=1}^{m} f(x_j)\, \ell_j(x)\, w(x)\, dx \\ &= \sum_{j=1}^{m} f(x_j) \int_a^b \ell_j(x)\, w(x)\, dx = \sum_{j=1}^{m} f(x_j)\, w_j = \mathcal{Q}[f]\,.\end{aligned}$$

Therefore the precision of quadrature formula (1.1) is at least $m-1$. $\square$

3. Maximum Possible Precision. The second fact we establish is that the precision of formula (1.1) can be at most $2m-1$. This is the expected result based upon counting. The precision of formula (1.1) will be $2m-1$ if and only if

$$\mathcal{Q}[x^k] = \mathcal{I}[x^k] \qquad \text{for every } k \in \{0, 1, \cdots, 2m-1\}\,.$$

This gives a system of $2m$ algebraic equations that must be satisfied by the $2m$ unkowns $\{x_j\}_{j=1}^m$ and $\{w_j\}_{j=1}^m$. This system takes the form

$$\sum_{j=1}^m x_j^k\, w_j = \int_a^b x^k\, w(x)\, dx \qquad \text{for every } k \in \{0, 1, \cdots, 2m-1\}\,. \tag{3.1}$$

While this system is linear in the $\{w_j\}_{j=1}^m$, it is nonlinear in the $\{x_j\}_{j=1}^m$. It is not obvious that this system generally has a solution, or that a solution would be unique if it exists. Rather than attack system (3.1) directly, we prove the following.

Fact 2. The precision of quadrature formula (1.1) can be at most $2m-1$.

Proof. Let $p(x)$ be the polynomial of degree m defined by

$$p(x) = \prod_{k=1}^m (x - x_k)\,. \tag{3.2}$$

It is clear that

$$\mathcal{Q}[p^2] = \sum_{j=1}^m p(x_j)^2\, w_j = \sum_{j=1}^m \prod_{k=1}^m (x_j - x_k)^2\, w_j = 0\,,$$

$$\mathcal{I}[p^2] = \int_a^b p(x)^2\, w(x)\, dx = \int_a^b \prod_{k=1}^m (x - x_k)^2\, w(x)\, dx > 0\,.$$

Because the degree of p^2 is $2m$, if the precision of formula (1.1) is greater than $2m-1$ then we must have $\mathcal{I}[p^2] = \mathcal{Q}[p^2]$, which contradicts the above relations. Therefore the precision of quadrature formula (1.1) can be at most $2m-1$. □

The polynomial $p(x)$ defined by (3.2) plays a central role in what follows. We dub it the *points polynomial* because its roots are the quadrature points.

4. Precision Orthogonality Condition. The third fact we establish characterizes the precision of quadrature formula (1.1) by an orthogonality condition on the points polynomial $p(x)$ that was defined by (3.2). This characterization was first given by Jacobi and underlies all Gauss quadrature methods.

Fact 3. Let $k \in \{0, \cdots, m-1\}$. Then the precision of quadrature formula (1.1) is $m+k$ if and only if $\{w_j\}_{j=1}^m$ is given by (2.1) and the points polynomial $p(x)$ defined by (3.2) satisfies the orthogonality condition

$$\int_a^b p(x)\, q(x)\, w(x)\, dx = 0 \qquad \text{for every } q \in \Pi^k\,, \tag{4.1}$$

where Π^k denotes the linear space of all polynomials with degree at most k.

Remark. Condition (4.1) is called an *orthogonality condition* because it asserts that the polynomial $p(x)$ is orthogonal to the linear space Π^k with respect to the scalar product

$$\langle q \,|\, r \rangle = \int_a^b q(x)\, r(x)\, w(x)\, dx = \mathcal{I}[q\, r]\,. \tag{4.2}$$

Proof. ($\Rightarrow$) Because the precision of formula (1.1) is $m+k > m-1$, Fact 1 implies that $\{w_j\}_{j=1}^m$ is given by (2.1).

Let $q \in \Pi^k$. Then $f(x) = p(x)\, q(x)$ is a polynomial of degree at most $m+k$. Because the precision of formula (1.1) is $m+k$, we have

$$\int_a^b p(x)\, q(x)\, w(x)\, dx = \mathcal{I}[p\, q] = \mathcal{Q}[p\, q] = \sum_{j=1}^m p(x_j)\, q(x_j)\, w_j = 0\,.$$

Therefore condition (4.1) holds.

($\Leftarrow$) Let $f(x)$ be a polynomial of degree at most $m+k$. Then there exist unique polynomials $q \in \Pi^k$ and $r \in \Pi^{m-1}$ such that

$$f(x) = p(x)\, q(x) + r(x)\,.$$

By the orthogonality condition (4.2) we see that

$$\mathcal{I}[f] = \int_a^b p(x)\, q(x)\, w(x)\, dx + \int_a^b r(x)\, w(x)\, dx = \int_a^b r(x)\, w(x)\, dx = \mathcal{I}[r]\,.$$

Because $p(x_j) = 0$ for every $j \in \{1, \cdots, m\}$, we see that

$$\mathcal{Q}[f] = \sum_{j=1}^m p(x_j)\, q(x_j)\, w_j + \sum_{j=1}^m r(x_j)\, w_j = \sum_{j=1}^m r(x_j)\, w_j = \mathcal{Q}[r]\,.$$

Because $\{w_j\}_{j=1}^m$ are given by (2.1), Fact 1 implies $\mathcal{I}[r] = \mathcal{Q}[r]$. The foregoing calculations thereby show that $\mathcal{I}[f] = \mathcal{Q}[f]$. Therefore quadrature formula (1.1) has precision $m+k$. □

5. Orthogonal Polynomials. The points polynomial $p(x)$ defined by (3.2) is the unique monic polynomial of degree m with m simple roots given by the quadrature points $\{x_j\}_{j=1}^m$. Here we show that for each m there is a unique monic polynomial $p_m(x)$ of degree m that satisfies the orthogonality condition (4.1) with $k = m-1$. We also show that this polynomial has m simple roots.

Fact 4. Let $\{p_n(x)\}_{n=0}^\infty$ be the sequence of polynomials given by

$$p_0(x) = 1\,, \qquad\qquad p_1(x) = x - \frac{\mathcal{I}[x]}{\mathcal{I}[1]}\,, \tag{5.1a}$$

$$p_n(x) = \left(x - \frac{\mathcal{I}[x\,p_{n-1}^2]}{\mathcal{I}[p_{n-1}^2]}\right) p_{n-1}(x) - \frac{\mathcal{I}[p_{n-1}^2]}{\mathcal{I}[p_{n-2}^2]}\, p_{n-2}(x) \qquad \text{for every } n > 1\,. \tag{5.1b}$$

Then $\{p_n(x)\}_{n=0}^\infty$ satisfy the orthogonality relations

$$\langle p_m \,|\, p_n \rangle = 0 \qquad \text{for every } m, n \in \mathbb{N} \text{ with } m \neq n\,. \tag{5.2}$$

Moreover, for every $n \in \mathbb{N}$ we have the spanning property

$$\Pi^n = \operatorname{Span}\{p_0(x)\,,\, p_1(x)\,,\, \cdots\,,\, p_n(x)\}\,. \tag{5.3}$$

Finally, for every $m \in \mathbb{Z}_+$ the polynomial $p_m(x)$ is the unique monic polynomial of degree m that satisfies the orthogonality condition (4.1) with $k = m-1$.

Remark. Algorithm (5.1) generates the orthogonal monic polynomials $\{p_n(x)\}_{n=0}^\infty$ from the monomials $\{x^n\}_{n=0}^\infty$. It is related to the Gram-Schmidt orthonormalization algorithm, which generates orthonormal polynomials from the monomials.

Proof. The orthogonality relations (5.2), the spanning property (5.3), and the fact $p_m(x)$ is a monic polynomial of degree m can be proved by induction. Those proofs are left as exercises. Here we only prove the final assertions.

Let $m \in \mathbb{Z}_+$. Let $q \in \Pi^{m-1}$. The spanning property (5.3) implies that there exists constants $\{c_n\}_{n=0}^{m-1}$ such that

$$q(x) = \sum_{n=0}^{m-1} c_n p_n(x)\,.$$

The orthogonality relations (5.2) then imply

$$\int_a^b p_m(x)\, q(x)\, w(x)\, dx = \sum_{n=0}^{m-1} c_n \int_a^b p_m(x)\, p_n(x)\, w(x)\, dx = 0\,.$$

Therefore $p_m(x)$ satisfies orthogonality condition (4.1) with $k = m-1$.

Now let $p(x)$ be any monic polynomial of degree m that satisfies orthogonality condition (4.1) with $k = m - 1$. We want to show that $p(x) = p_m(x)$. Let $r(x) = p(x) - p_m(x)$. Because $p(x)$ and $p_m(x)$ both are monic polynomials of degree m, $r(x)$ is a polynomial of degree at most $m - 1$. Then because $p(x)$ and $p_m(x)$ both satisfy orthogonality condition (4.1) with $k = m - 1$, we see that

$$\begin{aligned}\int_a^b r(x)^2\, w(x)\, dx &= \int_a^b \big(p(x) - p_m(x)\big)\, r(x)\, w(x)\, dx \\ &= \int_a^b p(x)\, r(x)\, w(x)\, dx - \int_a^b p_m(x)\, r(x)\, w(x)\, dx = 0\,.\end{aligned}$$

This implies that $r(x) = 0$, whereby $p(x) = p_m(x)$. Therefore $p_m(x)$ is the unique monic polynomial of degree m that satisfies the orthogonality condition (4.1) with $k = m - 1$. □

The following fact shows that every root of the orthogonal polynomials given by (5.1) is simple and lies in (a, b). This means that for every $m \in \mathbb{Z}_+$ the polynomial $p_m(x)$ is the point polynomial of a Gauss quadrature method.

Fact 5. Let $\{p_n(x)\}_{n=0}^{\infty}$ be the sequence of polynomials given by (5.1). For every $n \geq 1$ the polynomial $p_n(x)$ has n simple real roots that all lie in (a, b) and strictly interlace with the $n - 1$ simple roots of $p_{n-1}(x)$. This means that if $p_{n-1}(x)$ has real roots $\{x_j^{(n-1)}\}_{j=1}^{n-1}$ that satisfy

$$a < x_1^{(n-1)} < x_2^{(n-1)} < \cdots < x_{n-2}^{(n-1)} < x_{n-1}^{(n-1)} < b\,, \tag{5.4}$$

then $p_n(x)$ has real roots $\{x_j^{(n)}\}_{j=1}^{n}$ that have the strict interlacing property

$$\begin{aligned} a < x_1^{(n)} < x_1^{(n-1)} < x_2^{(n)} < x_2^{(n-1)} &< \cdots \\ \cdots < x_{n-2}^{(n-1)} < x_{n-1}^{(n)} < x_{n-1}^{(n-1)} &< x_n^{(n)} < b\,. \end{aligned} \tag{5.5}$$

Proof. Here we will only prove that the roots are simple and lie in (a, b). The strict interlacing property (5.5) can be proved by induction. That proof is left as an exercise.

Let $n \in \mathbb{N}$. Let $\{x_j\}_{j=1}^{n}$ be the roots of $p_n(x)$, so that

$$p_n(x) = \prod_{j=1}^{n} (x - x_j)\,.$$

Let $J \subset \{1, \cdots, n\}$ be a list of indices such that $\{x_j\}_{j \in J}$ is the set of all roots of $p_n(x)$ that lie in (a, b) and have odd multiplicity with each such root listed once. Then

$$q(x) = \prod_{j \in J} (x - x_j) \text{ has only simple roots that lie in } (a, b)\,,$$

$$p_n(x)\, q(x) \text{ does not change sign over } (a, b)\,.$$

Because $p_n(x)\,q(x)$ does not change sign over (a,b) we have

$$\langle p_n \mid q \rangle = \int_a^b p_n(x)\,q(x)\,w(x)\,dx \neq 0\,.$$

However, if $p_n(x)$ either has a root outside (a,b) or has a root inside (a,b) that is not simple then $q \in \Pi^{n-1}$, whereby the above fact contradicts the orthogonality condition (4.2) that was asserted in Fact 4. Therefore $p_n(x)$ has n simple roots that all lie in (a,b). □

6. Gauss Quadrature Sets. We now ready construct quadrature points and weights for which the quadrature formula (1.1) has precision $2m-1$ from the sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$.

> **Fact 6.** Let $\{p_n(x)\}_{n=0}^{\infty}$ be the sequence of polynomials given by (5.1). For each $m \in \mathbb{Z}_+$ we let $\{x_j\}_{j=1}^{m}$ be the roots of $p_m(x)$ and let $\{w_j\}_{j=1}^{m}$ be given by (2.1). Then the quadrature formula (1.1) has precision $2m-1$.

Remark. Fact 5 insures that $p_m(x)$ has m simple roots. When m is small they can be found analytically. When m is not small they generally have to be found numerically. Some orthogonal polynomials have roots that can be found analytically for every m.

Proof. Fact 4 implies that $p_m(x)$ satisfies the orthogonality condition (4.1) with $k = m-1$. Then Fact 3 implies that the quadrature formula (1.1) has precision $2m-1$. □

Remark. Quadrature sets constructed as in Fact 6 are called Gauss quadrature sets.

> **Fact 7.** Let $\{x_j\}_{j=1}^{m}$ and $\{w_j\}_{j=1}^{m}$ be a Gauss quadrature set. Then $w_j > 0$ for every $j \in \{1, \cdots, m\}$.

Remark. This result contrasts with the fact that quadrature weights that are constructed from uniformly spaced quadrature points by the Newton-Cotes recipe can be negative.

Proof. Let $j \in \{1, \cdots, m\}$. Let $\ell_j(x)$ be given by (2.2). Because $\ell_j(x)$ has degree $m-1$, $\ell_j(x)^2$ has degree $2m-2$. Because $\mathcal{Q}$ has precision $2m-1$ we have $\mathcal{Q}[\ell_j^2] = \mathcal{I}[\ell_j^2]$. Hence,

$$w_j = \sum_{k=1}^{m} \ell_j(x_k)^2\,w_k = \mathcal{Q}[\ell_j^2] = \mathcal{I}[\ell_j^2] = \int_a^b \ell_j(x)^2\,w(x)\,dx > 0\,.$$

Therefore $w_j > 0$ for every $j \in \{1, \cdots, m\}$. □

Remark. Given the roots $\{x_j\}_{j=1}^{m}$ of $p_m(x)$, the best way to compute the weights $\{w_j\}_{j=1}^{m}$ is by directly solving the linear system $\mathcal{Q}[x^k] = \mathcal{I}[x^k]$ with $k = 0, \cdots, m-1$ rather than by using (2.1). This approach is used in the examples below.

Example 1. The classical example featured in the textbook are the Gauss quadrature methods for definite integrals in the form

$$\mathcal{I}[f] = \int_{-1}^{1} f(x)\, dx\,. \tag{6.1}$$

This is (1.2) for the case $(a, b) = (-1, 1)$ and $w(x) = 1$.

The orthogonal polynomials for this case can be generated either by formula (5.1) or by a direct construction. The first six are

$$\begin{aligned} p_0(x) &= 1\,, & p_1(x) &= x\,, \\ p_2(x) &= x^2 - \tfrac{1}{3}\,, & p_3(x) &= x^3 - \tfrac{3}{5}x\,, \\ p_4(x) &= x^4 - \tfrac{6}{7}x^2 + \tfrac{3}{35}\,, & p_5(x) &= x^5 - \tfrac{10}{9}x^3 + \tfrac{5}{21}x\,. \end{aligned}$$

These are the Legendre polynomials. Here they have been normalized to be monic. Other normalizations might change each of them by a multiplicitive factor. Because we need only the roots of these polynomials, these factors do not matter.

Because $p_1(x)$ has root $\{0\}$, the Gauss quadrature point for $m = 1$ is $x_1 = 0$. The associated quadrature weight w_1 is determined from the relation

$$1 \cdot w_1 = \mathcal{Q}[1] = \mathcal{I}[1] = 2\,.$$

We see that the Gauss quadrature weight for $m = 1$ is $w_1 = 2$. This is just the midpoint rule.

Because $p_2(x)$ has roots $\{-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\}$, the Gauss quadrature points for $m = 2$ are

$$x_1 = -\sqrt{\tfrac{1}{3}}\,, \qquad x_2 = \sqrt{\tfrac{1}{3}}\,.$$

The associated quadrature weights are determined from the relations

$$\begin{aligned} 1 \cdot w_1 + 1 \cdot w_2 &= \mathcal{Q}[1] = \mathcal{I}[1] = 2\,, \\ -\sqrt{\tfrac{1}{3}} \cdot w_1 + \sqrt{\tfrac{1}{3}} \cdot w_2 &= \mathcal{Q}[x] = \mathcal{I}[x] = 0\,. \end{aligned}$$

We see that the Gauss quadrature weights for $m = 2$ are

$$w_1 = 1\,, \qquad w_2 = 1\,.$$

Because $p_3(x)$ has roots $\{-\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}\}$, the Gauss quadrature points for $m = 3$ are

$$x_1 = -\sqrt{\tfrac{3}{5}}\,, \qquad x_2 = 0\,, \qquad x_3 = \sqrt{\tfrac{1}{3}}\,.$$

The associated quadrature weights are determined from the relations

$$\begin{aligned} 1 \cdot w_1 + 1 \cdot w_2 + 1 \cdot w_3 &= \mathcal{Q}[1] = \mathcal{I}[1] = 2\,, \\ -\sqrt{\tfrac{3}{5}} \cdot w_1 + 0 \cdot w_2 + \sqrt{\tfrac{3}{5}} \cdot w_3 &= \mathcal{Q}[x] = \mathcal{I}[x] = 0\,, \\ \tfrac{3}{5} \cdot w_1 + 0 \cdot w_2 + \tfrac{3}{5} \cdot w_3 &= \mathcal{Q}[x^2] = \mathcal{I}[x^2] = \tfrac{2}{3}\,. \end{aligned}$$

We find that the Gauss quadrature weights for $m = 3$ are

$$w_1 = \tfrac{5}{9}\,, \qquad w_2 = \tfrac{8}{9}\,, \qquad w_3 = \tfrac{5}{9}\,.$$

Example 2. Consider Gauss quadrature methods for definite integrals in the form

$$\mathcal{I}[f] = \int_0^2 f(x)\, x\, dx\,. \tag{6.2}$$

This is (1.2) for the case $(a, b) = (0, 1)$ and $w(x) = x$.

The orthogonal polynomials for this case can be generated either by formula (5.1) or by a direct construction. The first three are

$$p_0(x) = 1\,, \qquad p_1(x) = x - \tfrac{4}{3}\,, \qquad p_2(x) = x^2 - \tfrac{12}{5}x + \tfrac{6}{5}\,.$$

These polynomials do not have a commonly used name.

Because $p_1(x)$ has root $\{\frac{4}{3}\}$, the Gauss quadrature point for $m = 1$ is $x_1 = \frac{4}{3}$. The associated quadrature weight is determined from the relation

$$1 \cdot w_1 = \mathcal{Q}[1] = \mathcal{I}[1] = 2\,.$$

We see that the Gauss quadrature weight for $m = 1$ is $w_1 = 2$.

Because $p_2(x)$ has roots $\{\frac{6}{5} - \frac{\sqrt{6}}{5}, \frac{6}{5} + \frac{\sqrt{6}}{5}\}$, the Gauss quadrature points for $m = 2$ are

$$x_1 = \tfrac{6}{5} - \tfrac{\sqrt{6}}{5}\,, \qquad x_2 = \tfrac{6}{5} + \tfrac{\sqrt{6}}{5}\,.$$

The associated quadrature weights are determined from the relations

$$\begin{aligned} 1 \cdot w_1 + 1 \cdot w_2 &= \mathcal{Q}[1] = \mathcal{I}[1] = 2\,, \\ (\tfrac{6}{5} - \tfrac{\sqrt{6}}{5}) \cdot w_1 + (\tfrac{6}{5} + \tfrac{\sqrt{6}}{5}) \cdot w_2 &= \mathcal{Q}[x] = \mathcal{I}[x] = \tfrac{8}{3}\,. \end{aligned}$$

We find that the Gauss quadrature weights for $m = 2$ are

$$w_1 = 1 - \tfrac{\sqrt{6}}{9}\,, \qquad w_2 = 1 + \tfrac{\sqrt{6}}{9}\,.$$

Example 3. Consider Gauss quadrature methods for definite integrals in the form

$$\mathcal{I}[f] = \int_0^\infty f(x)\, e^{-x} dx\,. \tag{6.2}$$

This is (1.2) for the case $(a, b) = (0, \infty)$ and $w(x) = e^{-x}$.

The orthogonal polynomials for this case can be generated either by formula (5.1) or by a direct construction. The first four are

$$\begin{aligned} p_0(x) &= 1\,, & p_1(x) &= x - 1\,, \\ p_2(x) &= x^2 - 4x + 2\,, & p_3(x) &= x^3 - 9x^2 + 18x - 6\,. \end{aligned}$$

These are the Laguerre polynomials. Here they have been normalized to be monic. Other normalizations might change each of them by a multiplicitive factor. Because we need only the roots of these polynomials, these factors do not matter.

Because $p_1(x)$ has root $\{1\}$, the Gauss quadrature point for $m = 1$ is $x_1 = 1$. The associated quadrature weight is determined from the relation

$$1 \cdot w_1 = \mathcal{Q}[1] = \mathcal{I}[1] = 1\,.$$

We see that the Gauss quadrature weight for $m = 1$ is $w_1 = 1$.

Because $p_2(x)$ has roots $\{2 - \sqrt{2}, 2 + \sqrt{2}\}$, the Gauss quadrature points for $m = 2$ are

$$x_1 = 2 - \sqrt{2}\,, \qquad x_2 = 2 + \sqrt{2}\,.$$

The associated quadrature weights are determined from the relations

$$\begin{aligned} 1 \cdot w_1 + 1 \cdot w_2 &= \mathcal{Q}[1] = \mathcal{I}[1] = 1\,, \\ (2 - \sqrt{2}) \cdot w_1 + (2 + \sqrt{2}) \cdot w_2 &= \mathcal{Q}[x] = \mathcal{I}[x] = 1\,. \end{aligned}$$

We find that the Gauss quadrature weights for $m = 2$ are

$$w_1 = \tfrac{1}{2} + \tfrac{\sqrt{2}}{4}\,, \qquad w_2 = \tfrac{1}{2} - \tfrac{\sqrt{2}}{4}\,.$$