Riccati Equations Arising from Hydrodynamics

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presented *8 February 2016* in the *Applied PDE RIT*,
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Introduction

Incompressible (low Mach number) fluid flows are described by initial-value problems for a Navier-Stokes system that take the form

$$\nabla_x \cdot u = 0\,, \tag{1a}$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u\,, \tag{1b}$$

$$u(x, 0) = u_o(x)\,. \tag{1c}$$

Of course, the spatial domain and appropriate boundary conditions must also be specified. This system was introduced by Navier (1823). It built upon the system introduced by Euler (1757), which is obtained by setting $\nu = 0$ in (1b). For many such problems a unique classical solution is known to exist for a finite time whenever the initial data u_o is sufficiently regular. For $D > 2$ the solutions of these systems can develop velocity gradient concentrations. It is unkown if these concentrations can blow-up. In this talk we explore models of this potential blow-up.

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Illustrative Example: Linear Burgers Solutions

Before studying the INS system, let us consider the initial-value problem for the one-dimensional Burgers equation, about which more is known,

$$\partial_t u + u \partial_x u = \nu \partial_{xx} u \, , \qquad u(x, 0) = u_o(x) \, . \tag{2}$$

We start by trying to solve this problem for linear initial data

$$u_o(x) = a_o x + b_o \, . \tag{3}$$

We seek a solution of (2) in the linear form

$$u(x, t) = a(t) x + b(t) \, . \tag{4}$$

When the linear form (4) is placed into (2) we see that

$$\left(\dot{a} + a^2\right) x + \left(\dot{b} + ab\right) = 0 \,,$$

which along with (3) implies that $a(t)$ and $b(t)$ satisfy

$$\dot{a} + a^2 = 0 \,, \qquad a(0) = a_o \,, \tag{5a}$$

$$\dot{b} + ab = 0 \,, \qquad b(0) = b_o \,. \tag{5b}$$

Upon solving (5) we obtain

$$a(t) = \frac{a_o}{1 + ta_o} \,, \qquad b(t) = \frac{b_o}{1 + ta_o} \,.$$

Hence, there is a unique solution of the Burgers initial-value problem (2, 3) in the linear form (4).

This linear solution of the Burgers equation (2) is given by

$$u(x,t) = \frac{a_o x + b_o}{1 + t a_o}\,. \tag{6}$$

- If $a_o > 0$ then this solution exists for $-1/a_o < t < \infty$.

- If $a_o = 0$ then this solution exists for all t.

- If $a_o < 0$ then this solution exists for $-\infty < t < -1/a_o$.

While these are not finite energy solutions of (2), they capture key aspects of the behavior of every finite energy solution. Specifically, the case $a_o > 0$ describes *rarefaction*, while the case $a_o < 0$ describes *shock formation*.

Finite energy solutions of the Burgers equation will not develop an infinite spatial derivative in finite time. Rather, when ν is small these solutions will develop steep smooth shock profiles where dissipation will concentrate. The development of these smooth steep shock profiles is approximated by the linear solution (6) with $a_o = \partial_x u_o(x_o) < 0$ and $b_o = u_o(x_o)$, where the point x_o is a local minimizer of $\partial_x u_o(x)$.

As $\nu \to 0$ these solutions will approach a limit that is a weak solution of the *inviscid Burgers* equation (*Hopf* equation). In this limit the shock profiles will approach jump discontinuities (shocks) along which there is *anomalous dissipation*. The large-time asymptotics of the limiting weak solution will approach a combination of shocks and rarefactions. These rarefactions are approximated by the linear solution (6) with $a_o = \partial_x u_o(x_o) > 0$ and $b_o = u_o(x_o)$, where the point x_o is a local maximizer of $\partial_x u_o(x)$.

Linear Incompressible Navier-Stokes Solutions

Now let us apply this same approach to the INS system over $\mathbb{R}^D$ for $D \geq 2$:

$$\nabla_x \cdot u = 0\,, \tag{7a}$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u\,, \qquad u(x,0) = u_o(x)\,. \tag{7b}$$

We start by trying to solve this problem for linear initial data

$$u_o(x) = A_o x + b_o\,, \tag{8}$$

where $A_o \in \mathbb{R}^{D \times D}$, $\mathtt{tr}(A_o) = 0$, and $b_o \in \mathbb{R}^D$. (Here $\mathtt{tr}(M)$ denotes the trace of any square matrix M.) We seek a solution of (7) in the linear form

$$u(x,t) = A(t)x + b(t)\,. \tag{9}$$

When the linear form (9) is placed into (7) we see that

$$\begin{aligned} \mathrm{tr}(A) &= 0\,, \\ \left(\dot{A} + A^2\right) x + \left(\dot{b} + Ab\right) + \nabla_{\!x} p &= 0\,. \end{aligned}$$

This last equation can be solved for p if and only if

$$\dot{A} + A^2 \quad \text{is symmetric}\,,$$

in which case p is given by

$$p = -\tfrac{1}{2} x \cdot \left(\dot{A} + A^2\right) x - \left(\dot{b} + Ab\right) \cdot x + c\,, \tag{10}$$

where c is an arbitrary function of time. The arbitrary function c is expected because p only enters the INS system (7) through the $\nabla_{\!x} p$ term of the motion equation.

There are many choices of $A(t)$ and $b(t)$ that make $u(x,t)$ given by (9) a global solution of (7)! Indeed, this is done by every $A \in C^1([0,\infty),\mathbb{R}^{D\times D})$ and $b \in C^1([0,\infty),\mathbb{R}^D)$ such that

$$
\begin{aligned}
A(0) &= A_o\,, &\qquad b(0) &= b_o\,, \\
\mathrm{tr}\Big(A(t)\Big) &= 0\,, &\qquad \dot{A}(t) + A(t)^2 &\quad \text{is symmetric}\,.
\end{aligned}
\tag{11}
$$

Because $\mathtt{tr}(A_o) = 0$ the bottom two conditions can be combined as

$$
\dot{A}(t) + A(t)^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A(t)^2\Big)I \quad \text{is symmetric and traceless}\,.
$$

Just as easily we can construct classical solutions of (7) that blow-up at the endpoints of any given interval (t_L, t_R) containing 0!

Unlike for the Burgers equation, here the balances in the INS system do not determine a unique solution in the linear form (9)! This lack of uniqueness reflects the fact that the pressure p in the INS system depends upon u in a nonlocal way.

Conversely, let $g \in C([0,\infty),\mathbb{R}^D)$ and $H \in C([0,\infty),\mathbb{R}^{D\times D})$ such that $H(t)$ is symmetric and traceless for every $t \geq 0$. Let $A(t)$ and $b(t)$ satisfy

$$\dot{A} + A^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big)I = H(t)\,, \qquad A(0) = A_o\,, \tag{12a}$$

$$\dot{b} + Ab = g(t)\,, \qquad b(0) = b_o\,. \tag{12b}$$

Then $A(t)$ and $b(t)$ will satisfy (11) over the interval of definition (t_L, t_R) for the solution $A(t)$ of (12a). For every such g and H we have constructed a solution of the INS system (7) given by

$$\begin{aligned} u(x,t) &= A(t)x + b(t)\,, \\ p(x,t) &= c(t) - g(t)\cdot x - \tfrac{1}{2D}\,\mathrm{tr}\Big(A(t)^2\Big)|x|^2 - \tfrac{1}{2}x\cdot H(t)x\,. \end{aligned} \tag{13}$$

Remark. This solution will be unique if boundary conditions are imposed on the large x behavior of the pressure that select a unique g and H. It is not clear how to do this. For example, imposing a large x isotropy condition on $p(x,t)$ implies that $H(t) = 0$, but leaves the freedom to choose $g(t)$.

Models of Velocity Gradient Dynamics

Now consider the dynamics of the gradient tensor $A(x,t) = \nabla_x u(x,t)$ for an arbitrary solution $u(x,t)$ of the INS system (1). Taking the gradient of (1b) yields

$$\partial_t A + u \cdot \nabla_x A + A^2 + \nabla_x^2 p = \nu \Delta_x A \,, \qquad A(x,0) = \nabla_x u_o(x) \,. \tag{14}$$

Here $\nabla_x^2 p$ is the Hessian matrix of p, not its Laplacian! The divergence-free condition (1a) implies that $\mathrm{tr}(A(x,t)) = 0$, whereby taking the trace of (14) yields the so-called *pressure equation,*

$$-\Delta_x p = -\,\mathrm{tr}\Big(\nabla_x^2 p\Big) = \mathrm{tr}\Big(A^2\Big) \,. \tag{15}$$

This combined with (14) yields the *velocity gradient dynamics* equation

$$\begin{aligned} \partial_t A + u \cdot \nabla_x A + A^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big) I& \\ + \nabla_x^2 p - \tfrac{1}{D} \Delta_x p \, I = \nu \Delta_x A \,.& \end{aligned} \tag{16}$$

Patrick Vieillefosse (1982, 1984) proposed to model singularity formation in the three-dimensional incompressible Euler system — and therefore to model singularity or near singularity formation in the three-dimensional INS system by making the approximations

$$\nabla_x^2 p - \tfrac{1}{D}\Delta_x p\, I = 0\,, \qquad \nu\Delta_x A = 0\,. \tag{17}$$

The first states that the Hessian of p is nearly isotropic compared to the anisotropies that arise from A. The second states that the viscosity will have little effect on singularity formation.

The result is the so-called *Restricted Euler* (RE) equation

$$\partial_t A + u\cdot\nabla_x A + A^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big) I = 0\,. \tag{18}$$

Because this equation does not preserve the symmetry relationship on the three-tensor $\nabla_x A$ that is required when $A = \nabla_x u$ for some u, the approximation that $\nabla_x^2 p$ is isotropic is not generally valid globally.

Remark. The RE model was investigated by Liu and Tadmor (2002) by Liu, Tadmor, and Wei (2010), and by Wei (2011).

This model can be extended to include an anisotropic pressure Hessian. For example, we can consider the family of approximations given by

$$\begin{aligned}\nabla_x^2 p - \tfrac{1}{D}\Delta_x p\, I = \eta_1(A)\Big[A^T A + A A^T - \tfrac{2}{D}\,\mathrm{tr}\Big(A^T A\Big) I\Big] \\ + \eta_2(A)\Big[A^2 + A^{T2} - \tfrac{2}{D}\,\mathrm{tr}\Big(A^2\Big) I\Big] \\ + \eta_3(A)\Big[A^T A - A A^T\Big]\,,\end{aligned} \tag{19}$$

where $\eta_1(A)$, $\eta_2(A)$, and $\eta_3(A)$ are unitless scalar coefficients that can continuously depend upon unitless combinations of A.

When these approximations are placed into the velocity gradient dynamics equation (16) we obtain the so-called *Generalized Restricted Euler* (GRE) equation is

$$
\begin{aligned}
\partial_t A + u \cdot \nabla_x A + A^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big) I& \\
+\, \eta_1(A)\Big[A^T A + A A^T - \tfrac{2}{D}\,\mathrm{tr}\Big(A^T A\Big) I\Big]& \\
+\, \eta_2(A)\Big[A^2 + A^{T2} - \tfrac{2}{D}\,\mathrm{tr}\Big(A^2\Big) I\Big]& \\
+\, \eta_3(A)\Big[A^T A - A A^T\Big] = 0\,.&
\end{aligned}
\tag{20}
$$

Every member of this family preserves all of the dilational, rotational, and Galilean symmetries of the velocity gradient dynamics equation (16). See Meneveau (2011) for a review article.

One member of this family can be derived starting from the formula for $\nabla_x^2 p$ obtained from the pressure equation (15) for finite-energy Navier-Stokes solutions u. We then decompose $\nabla_x^2 p - \frac{1}{D} I \, \Delta_x p$ as

$$\nabla_x^2 p - \tfrac{1}{D} I \, \Delta_x p = \mathcal{L}[u] + \mathcal{N}[u] \,, \tag{21a}$$

where $\mathcal{L}[u]$ is the *local* approximation given by

$$\begin{aligned} \mathcal{L}[u] = &-\tfrac{2}{D(D+4)}\Big[\nabla_x^2 |u|^2 - \tfrac{1}{D} I \, \Delta_x |u|^2\Big] \\ &+ \tfrac{2D^2+4D-8}{D(D+2)(D+4)}\Big[\nabla_x \vee \Big(\nabla_x \cdot u^{\vee 2}\Big) - \tfrac{1}{D} I \, \nabla_x^2 : u^{\vee 2}\Big)\Big] \,, \end{aligned} \tag{21b}$$

and $\mathcal{N}[u]$ is the *nonlocal* correction given by

$$\mathcal{N}[u] = \text{F.P.} \int \nabla_x^4 g(x-y) : u^{\vee 2}(y) \, \mathrm{d}y \,, \tag{21c}$$

where F.P. indicates a *Hadamard finite part* integral and $g(x-y)$ is the whole space Green function for the Laplace operator.

If we neglect $\mathcal{N}[u]$ and use the approximation $\nabla_x u(x,t) \approx A(t)$ in $\mathcal{L}[u]$ then we obtain

$$\begin{aligned}\nabla_x^2 p - \tfrac{1}{D}\, I\, \Delta_x p \approx &-\tfrac{4}{D(D+4)}\Big[A^T A - \tfrac{1}{D} I \,\mathrm{tr}\Big(A^T A\Big)\Big] \\ &+ \tfrac{D^2+2D-4}{D(D+2)(D+4)}\Big[A^2 + A^{T2} - \tfrac{2}{D} I \,\mathrm{tr}\Big(A^2\Big)\Big] \,.\end{aligned}$$

When this approximation is placed into the velocity gradient equation (16) then we obtain the model

$$\begin{aligned}\partial_t A + u \cdot \nabla_x A + A^2 - \tfrac{1}{D} I \,\mathrm{tr}\Big(A^2\Big) & \\ + \beta\Big[A^T A - \tfrac{1}{D} I \,\mathrm{tr}\Big(A^T A\Big)\Big] & \\ + \gamma\Big[A^2 + A^{T2} - \tfrac{2}{D} I \,\mathrm{tr}\Big(A^2\Big)\Big] &= 0\,,\end{aligned} \tag{22a}$$

where

$$\beta = -\tfrac{4}{D(D+4)}\,, \qquad \gamma = \tfrac{D^2+2D-4}{D(D+2)(D+4)}\,. \tag{22b}$$

Model (22) is the member of the family of models (20) with

$$\eta_1 = \eta_3 = \tfrac{1}{2}\beta = -\tfrac{2}{D(D+4)}\,, \qquad \eta_2 = \gamma = \tfrac{D^2+2D-4}{D(D+2)(D+4)}\,.$$

The Vieillefosse model corresponds to (20) with $\eta_1 = \eta_2 = \eta_3 = 0$.

For $D = 2$ model (22) corresponds to (20) with

$$\eta_1 = \eta_3 = -\tfrac{1}{6}\,, \qquad \eta_2 = \tfrac{1}{12}\,.$$

This leaves the vorticity field constant and gives an oscillatory dynamics to the strain rate field. This contrasts sharply with the Vieillefosse model, which gives no dynamics for $D = 2$.

For $D = 3$ model (22) corresponds to (20) with

$$\eta_1 = \eta_3 = -\tfrac{2}{21}\,, \qquad \eta_2 = \tfrac{11}{105}\,.$$

We see that this model gives a 10% correction to the Vieillefosse model.

By decomposing $A = S + \Omega$ where $S^T = S$ and $\Omega^T = -\Omega$, the family of velocity gradient models (20) can be expressed as

$$\begin{aligned}\partial_t S + u \cdot \nabla_x S &+ 2\eta_1 \Big[S^2 - \Omega^2 - \tfrac{1}{D} \operatorname{tr}\Big(S^2 - \Omega^2\Big) I \Big] \\ &+ (1 + 2\eta_2) \Big[S^2 + \Omega^2 - \tfrac{1}{D} \operatorname{tr}\Big(S^2 + \Omega^2\Big) I \Big] \\ &+ 2\eta_3 \Big[S\Omega - \Omega S \Big] = 0 \,,\end{aligned} \tag{23a}$$

$$\partial_t \Omega + u \cdot \nabla_x \Omega + S\Omega + \Omega S = 0 \,. \tag{23b}$$

The symmetric matrix S relates to strain while the skew-symmetric matrix Ω relates to rotation. When $\eta_1 = \eta_1(S)$, $2\eta_1 = 1 + 2\eta_2$, and $\eta_3 = 0$, the first equation decouples as

$$\partial_t S + u \cdot \nabla_x S + 4\eta_1(S) \Big[S^2 - \tfrac{1}{D} \operatorname{tr}\Big(S^2\Big) I \Big] = 0 \,. \tag{24}$$

When $\eta_1 = \frac{1}{4}(1-\alpha^2)$, $\eta_2 = \frac{1}{4}(\alpha^2 - 1)$, and $\eta_3 = 0$ for some real α, we have the decoupled equation

$$\partial_t B + u \cdot \nabla_x B + B^2 - \frac{1}{D}\,\mathrm{tr}\Big(B^2\Big) I = 0\,, \tag{25}$$

where $B = S + \alpha\Omega$. When $\alpha = 1$ this becomes the Vieillefosse model (18). When $\alpha = 0$ this becomes the model (24) with $\eta_1 = \frac{1}{4}$.

Models (24) and (25) have spatially homogeneous solutions that satisfy a matrix Riccati equation of the form

$$\dot{B} + \mu(B)\left[B^2 - \frac{1}{D}\,\mathrm{tr}\Big(B^2\Big) I\right] = 0\,,$$

where $\mu(B) \neq 0$ is a unitless scalar coefficient. We will analyze solutions of this equation for the case $\mu(B) > 0$. The analogous analysis for the case $\mu(B) < 0$ is similar.

Solutions of Matrix Riccati Equations

Our main result concerns the evolution of a traceless $D \times D$ matrix $B(t)$ governed by a matrix Riccati initial-value problem in the form

$$\dot{B} + \mu(B)\left[B^2 - \frac{1}{D}\,\mathrm{tr}\Big(B^2\Big)I\right] = 0\,, \qquad B(0) = B_o\,, \tag{26}$$

where $\mathtt{tr}(B_o) = 0$ and $\mu(B) > 0$ is a unitless scalar coefficient that can continuously depend upon unitless combinations of B.

Stationary solutions of this problem are traceless matrices B that satisfy

$$B^2 - \frac{1}{D}\,\mathrm{tr}\Big(B^2\Big)\,I = 0\,.$$

A matrix B is a stationary solution if and only if it has one of the forms:

- $B = N$ where $N^2 = 0$ and $\mathtt{tr}(N) = 0$;
- $B = \beta R$ where $\beta > 0$, $R^2 = I$, and $\mathtt{tr}(R) = 0$;
- $B = \xi J$ where $\xi > 0$, $J^2 = -I$, and $\mathtt{tr}(J) = 0$.

The second and third cases can arise only when D is even.

- For the second case this follows because:
  - $R^2 = I$ implies that every eigenvalue of R is either -1 or 1;
  - $\mathtt{tr}(R) = 0$ implies that the eigenvalues sum to zero.

  Therefore D must be even and -1 and 1 each have multiplicity $\frac{D}{2}$.

- For the third case this follows because:
  - $J^2 = -I$ implies that every eigenvalue of J is either $-i$ or i;
  - $\mathtt{tr}(J) = 0$ implies that the eigenvalues sum to zero.

  Therefore D must be even and $-i$ and i each have multiplicity $\frac{D}{2}$.

This shows that there are many more stationary solutions when D is even.
In particular all traceless 2×2 matices are stationary solutions.

The original Vieillefosse model ($D = 3$ and $\mu(B) = 1$) was solved by Cantwell (1992), but his solution does not extend to $D > 3$. For $D > 3$ and $\mu(B) = 1$ the model was investigated by Liu and Tadmor (2002), by Liu, Tadmor, and Wei (2010), and by Wei (2011). They analyzed the dynamics of the eigenvalues of $B(t)$, a technique that they dubbed *spectral dynamics*. More structure emerged with each investigation. Eventually, enough conserved quantities were discovered to suggest that the problem was integrable in some sense.

Lax (1968) proposed that evolution equations with enough conserved quantities might be recast in terms of so-called *Lax pairs*. In this framework the conserved quantities are associated with the stationary spectrum of an operator or matrix that is evolving by a so-called *isospectral flow*. However, problem (26) has resisted all attempts to recast it in this framework.

This paper takes a different approach that gives an explicit sense in which initial-value problem (26) is integrable. Its main result is that the solution to (26) is given by $B(t) = \mathcal{B}(\tau(t))$ where $\mathcal{B}(\tau)$ is given by the explicit formula

$$\begin{aligned}\mathcal{B}(\tau) = {}& \det(I + \tau B_o)^{\frac{2}{D}} \\ & \left[(I + \tau B_o)^{-1} B_o - \frac{1}{D}\,\mathrm{tr}\Big((I + \tau B_o)^{-1} B_o\Big)\, I\right],\end{aligned} \tag{27a}$$

and the so-called artificial time $\tau(t)$ solves the initial-value problem

$$\dot{\tau} = \mu(\mathcal{B}(\tau))\, \det(I + \tau B_o)^{\frac{2}{D}}, \qquad \tau(0) = 0\,. \tag{27b}$$

We can directly checking that this solves the initial-value problem (26). Let $B(t) = \mathcal{B}(\tau(t))$ where $\mathcal{B}(\tau)$ and $\tau(t)$ are given by (27). Because $\mathcal{B}(0) = B_o$ and $\tau(0) = 0$, we see $B(0) = \mathcal{B}(\tau(0)) = \mathcal{B}(0) = B_o$, whereby the initial condition of (26) is satisfied.

We next show that the evolution equation of (26) is also satisfied. Because $B(t) = \mathcal{B}(\tau(t))$, by the chain rule and the τ evolution equation in (27b) we have

$$\dot{B} = \dot{\tau}\mathcal{B}'(\tau) = \mu(\mathcal{B}(\tau))\,\mathrm{det}(I + \tau B_o)^{\frac{2}{D}}\mathcal{B}'(\tau)\,.$$

Therefore

$$\begin{aligned}
&\dot{B} + \mu(B)\left[B^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(B^2\Big)I\right] \\
&\quad = \dot{\tau}\mathcal{B}'(\tau) + \mu(\mathcal{B}(\tau))\left[\mathcal{B}(\tau)^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(\mathcal{B}(\tau)^2\Big)\,I\right] \\
&\quad = \mu(\mathcal{B}(\tau))\left[\mathrm{det}(I + \tau B_o)^{\frac{2}{D}}\mathcal{B}'(\tau) + \mathcal{B}(\tau)^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(\mathcal{B}(\tau)^2\Big)\,I\right]\,.
\end{aligned} \tag{28}$$

Because $\mu(\mathcal{B}(\tau)) > 0$, the evolution equation of (26) is satisfied if and only if the quantity inside the last set of square brackets above vanishes.

We now need two calculus identities that apply to differentiable functions of τ which take values $M(\tau)$ that are invertible $D \times D$ real matrices. The first, due to Liouville (1838), states that if $\det(M(\tau)) > 0$ then

$$\log\Big(\det(M(\tau))\Big)' = \operatorname{tr}\Big(M(\tau)^{-1}M'(\tau)\Big)\,. \tag{29a}$$

The second, which is elementary, states that

$$\Big(M(\tau)^{-1}\Big)' = -M(\tau)^{-1}M'(\tau)M(\tau)^{-1}\,. \tag{29b}$$

Here prime denotes differentiation with respect to τ. For τ sufficiently small we have $\det(I + \tau B_o) > 0$, so that we can apply the identities (29) to $M(\tau) = I + \tau B_o$, whereby formula (27a) yields

$$\det(I + \tau B_o)^{\frac{2}{D}}\mathcal{B}'(\tau) + \mathcal{B}(\tau)^2 - \tfrac{1}{D}\operatorname{tr}\Big(\mathcal{B}(\tau)^2\Big)\, I = 0\,. \tag{30}$$

But this shows that the quantity inside the last set of square brackets in (28) vanishes. Therefore the evolution equation of (26) is also satisfied, whereby recipe (27) gives the solution of the initial-value problem (26).

Analysis of Nonstationary Solutions

The initial-value problem (27b) satisfied by τ is

$$\dot{\tau} = \mu(\mathcal{B}(\tau))\,\mathrm{det}(I + \tau B_o)^{\frac{2}{D}}\,, \qquad \tau(0) = 0\,. \tag{31}$$

Because we have assumed that $\mu(\mathcal{B}(\tau)) > 0$, the stationary points of this evolution equation are given by $-1/\beta$, where β is a nonzero real eigenvalue of B_o. A phase-line analysis shows that as t increases the value of $\tau(t)$ will run from τ_{min} to τ_{max} where

$$\begin{aligned} \tau_{\mathrm{min}} &= \sup\left\{-\frac{1}{\beta}\ :\ \beta \in \mathrm{Sp}(B_o)\,,\ \beta > 0\right\}\,, \\ \tau_{\mathrm{max}} &= \inf\left\{-\frac{1}{\beta}\ :\ \beta \in \mathrm{Sp}(B_o)\,,\ \beta < 0\right\}\,. \end{aligned} \tag{32}$$

Recall that $\sup\{\emptyset\} = -\infty$ and $\inf\{\emptyset\} = \infty$, so these formulas cover the cases when B_o either has no postive eigenvalues or has no negative eigenvalues. Notice that in all cases $\tau_{\mathrm{min}} < 0 < \tau_{\mathrm{max}}$.

A phase-line analysis of the τ evolution equation shows the following.

- If $-\infty < \tau_{\text{min}}$ then $\tau(t)$ will reach τ_{min} in finite time as t decreases from 0 when the eigenvalue of B_o associated with τ_{min} has multiplicity at less than $\frac{D}{2}$, and will reach τ_{min} in infinite time otherwise.

- If $\tau_{\text{max}} < \infty$ then $\tau(t)$ will reach τ_{max} in finite time as t increases from 0 when the eigenvalue of B_o associated with τ_{max} has multiplicity at less than $\frac{D}{2}$, and will reach τ_{max} in infinite time otherwise.

These facts follow because the right-hand side of the evolution equation will vanish like either $(\tau - \tau_{\text{min}})^{\frac{2m}{D}}$ or $(\tau_{\text{max}} - \tau)^{\frac{2m}{D}}$ respectively where m is the multiplicity of the associated eigenvalue of B_o. Therefore $\tau(t)$ will reach the stationary point in finite time if and only if $\frac{2m}{D} < 1$.

If $-\infty < \tau_{\text{min}}$ and the eigenvalue of B_o associated with τ_{min} has multiplicity at less than $\frac{D}{2}$ then $B(t) = \mathcal{B}(\tau(t))$ becomes unbounded in finite time as t decreases.

If $-\infty < \tau_{\text{min}}$ and the eigenvalue of B_o associated with τ_{min} has multiplicity at least $\frac{D}{2}$ then $B(t) = \mathcal{B}(\tau(t))$ becomes unbounded as $t \searrow -\infty$.

If $\tau_{\text{max}} < \infty$ and the eigenvalue of B_o associated with τ_{min} has multiplicity at less than $\frac{D}{2}$ then $B(t) = \mathcal{B}(\tau(t))$ becomes unbounded in finite time as t increases.

If $\tau_{\text{max}} < \infty$ and the eigenvalue of B_o associated with τ_{min} has multiplicity at least $\frac{D}{2}$ then $B(t) = \mathcal{B}(\tau(t))$ becomes unbounded as $t \nearrow \infty$.

If $\tau_{\text{min}} = -\infty$ or $\tau_{\text{max}} = \infty$ then the story becomes more complicated. When $\det(B_o) \neq 0$ the story is the following. Define

$$B_\infty = \det(B_o)^{\frac{2}{D}} \left[\tfrac{1}{D} \operatorname{tr}\!\left(B_o^{-1}\right) I - B_o^{-1} \right] .$$

If $\tau_{\text{min}} = -\infty$ then $B(t) = \mathcal{B}(\tau(t)) \to B_\infty$ in finite time as t decreases.

If $\tau_{\text{max}} = \infty$ then $B(t) = \mathcal{B}(\tau(t)) \to B_\infty$ in finite time as t increases.

In particular, if B_o has no real eigenvalues then $B(t)$ is periodic! This includes every solution in some neighborhood of each stationary solution in the form ξJ for some $\xi > 0$ and matrix J with $J^2 = -I$ and $\operatorname{tr}(J) = 0$. Notice that such matrices can only exist when D is even.

If $\tau_{\mathsf{min}} = -\infty$ or $\tau_{\mathsf{max}} = \infty$ then to tell the story when $\mathsf{det}(B_o) = 0$ we need an identity. If $\tau \neq 0$ and $-\frac{1}{\tau} \notin \mathsf{Sp}(B_o)$ then $\mathcal{B}(\tau)$ defined by formula (27a) can be expressed as

$$\begin{aligned}\mathcal{B}(\tau) = &\mathsf{det}\Big(\tfrac{1}{\tau}I + B_o\Big)^{\frac{2}{D}-1} \\ &\Big[\frac{1}{D}\,\mathsf{tr}\Big(\mathsf{Cof}\Big(\tfrac{1}{\tau}I + B_o\Big)\Big)\,I - \mathsf{Cof}\Big(\tfrac{1}{\tau}I + B_o\Big)^{T}\Big]\,.\end{aligned} \tag{33}$$

Here $\mathsf{Cof}(M)$ denotes the cofactor matrix of any square matrix M.

If $\tau_{\mathsf{min}} = -\infty$, $\mathsf{det}(B_o) = 0$, and $\mathsf{Cof}(B_o) \neq 0$ then $B(t) = \mathcal{B}(\tau(t))$ becomes unbounded in finite time as t decreases.

If $\tau_{\mathsf{max}} = \infty$, $\mathsf{det}(B_o) = 0$, and $\mathsf{Cof}(B_o) \neq 0$ then $B(t) = \mathcal{B}(\tau(t))$ becomes unbounded in finite time as t increases.

When $\eta_1 = \frac{1}{4}$, $\eta_2 = -\frac{1}{4}$, $\eta_3 = 0$ the matrix Riccati system has solution

$$\begin{aligned} S(t) &= \det(I + \tau S_o)^{\frac{2}{D}} \\ &\qquad \Big[(I + \tau S_o)^{-1} S_o - \tfrac{1}{D}\,\mathrm{tr}\Big((I + \tau S_o)^{-1} S_o\Big)\, I\Big]\,, \\ \Omega(t) &= \det(I + \tau S_o)^{\frac{1}{D}} (I + \tau S_o)^{-\frac{1}{2}} \Omega_0 (I + \tau S_o)^{-\frac{1}{2}}\,. \end{aligned} \tag{34a}$$

where $\tau(t)$ is the solution of the scalar initial-value problem

$$\dot{\tau} = \det(I + \tau S_o)^{\frac{2}{D}}\,, \qquad \tau(0) = 0\,. \tag{34b}$$

We remark that $I + \tau S_o$ is positive definite, so its square root is too.

Because S_o is symmetric and traceless, it has real eigenvalues that sum to zero. Hence, if $S_o \neq 0$ then it must have a positive and a negative eigenvalue. In that case $-\infty < \tau_{\mathrm{min}}$ and $\tau_{\mathrm{max}} < \infty$.

- $\tau(t)$ will reach τ_{min} in a finite time $t_L < 0$ as t decreases from 0 unless the eigenvalue of S_o associated with τ_{min} has multiplicity $\geq \frac{D}{2}$.

- $\tau(t)$ will reach τ_{max} in finite time $t_R > 0$ as t increases from 0 unless the eigenvalue of S_o associated with τ_{max} has multiplicity $\geq \frac{D}{2}$.

- Blow-ups of $S(t)$ behave like $(t_R - t)^{-1}$ as $t \nearrow t_R$ and like $(t - t_L)^{-1}$ as $t \searrow t_L$.

- $S(t)$ is eternal if and only if S_o has exactly two real eigenvalues, each with multiplicity $= \frac{D}{2}$. This can only occur when D is even and the eigenvalues are negatives of each other, whereby it is a stationary solution.

A few comments about $D = 3$.

- The vorticity will nearly align with the eigendirection associated with the most positive eigenvalue of S_o. This is consistent with what is reported by Ashurst, Kerstein, Kerr, and Gibson (1987) as being seen in simulations during vorticity stengthening.

- The blow-up of these solutions can be arrested by the inclusion of sufficient damping in the dynamical equation as a model for viscosity. There are analytic solutions in that case too.

- There is an entire family of similar models presented in a recent review by Charles Meneveau (2011).

Concluding Remarks

- There is a family of matrix Riccati equations that arise from hydrodynamics. The connection of these to general fluid flows is not clear.

- There are nearly explicit solutions for a subfamily of this family. Odd and even dimensions show very different behavior.

- Different models of velocity gradient dynamics lead to matrix Riccati equations with very different behavior.

Thank You!

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