Sample Problems for the Math 151a Midterm Exam
Professor Levermore, Fall 2014

(1) Use Gaussian elimination with backward substitution to solve the linear system

$$\begin{aligned} x_1 + x_2 \qquad\quad + x_4 &= 2\,, \\ 2x_1 + x_2 - x_3 + x_4 &= 1\,, \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4\,, \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3\,. \end{aligned}$$

(2) Consider the linear system

$$\begin{aligned} x_1 - x_2 + x_3 &= 0\,, \\ 12x_2 - x_3 &= 4\,, \\ 2x_1 + x_2 + x_3 &= 7\,. \end{aligned}$$

Find the row interchanges required to solve this system using Gaussian elimination
(a) with partial pivoting,
(b) with scaled partial pivoting.
(Do not use backward substitution to solve the system!)

(3) Let

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 2 & 0 & 5 \\ 2 & 1 & 6 \end{pmatrix}\,.$$

Find the factorization $A = LU$ where L is a lower trianglar matrix with ones on its diagonal and U is an upper triangular matrix.

(4) Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & a & -1 \\ 0 & -1 & 2 \end{pmatrix}\,.$$

(a) Find all $a \in \mathbb{R}$ for which A is invertible.
(b) Find all $a \in \mathbb{R}$ for which A is strictly diagonally dominant.
(c) Find all $a \in \mathbb{R}$ for which A is positive definite.
(d) Find all $a \in \mathbb{R}$ for which A has a factorization $A = LDL^T$ where L is a lower trianglar matrix with ones on its diagonal and D is a diagonal matrix with nonzero diagonal entries. (You do not have to find the factorization.)

(5) Consider the arithmetic expressions

$$\left(\tfrac{3}{20} + \tfrac{4}{11}\right) - \tfrac{1}{3}\,, \qquad \tfrac{3}{20} + \left(\tfrac{4}{11} - \tfrac{1}{3}\right).$$

In exact arithmetic both of these expressions are equal to $\frac{119}{660} = .180\overline{303}$.
(a) Use three-digit chopping decimal arithmetic to evaluate each of these expressions. Use the fact that $\frac{3}{20} = .15$, $\frac{4}{11} = .3636\overline{36}$, and $\frac{1}{3} = .3333\overline{33}$.
(b) Which of these evaluations gives the smallest error? Why?

(6) Let $f(x) = 4^x - 6x^2$.

(a) Prove that f has at least one zero in $[0, 1]$.

(b) Find p_1 and p_2 of the bisection method. Write your answers in the blank entries of the following table.

n	a_n	b_n	p_n	$f(p_n)$
1	0	1		
2				XXXX

(c) For what value of n will you achieve an accuracy of 10^{-3}?

(7) Let f be a continuous function over $[0, 2]$ such that

x	0	.25	.5	.75	1.	1.25	1.5	1.75	2.0
$f(x)$	1.39	1.11	.85	.43	.12	$-.23$	$-.56$	$-.97$	-1.34

(a) Prove that f has at least one zero in $[0, 2]$.

(b) Find p_0, p_1, and p_2 of the bisection method. Write your answers in the blank entries of the following table.

n	a_n	b_n	p_n	$f(p_n)$
0	0	2		
1				
2				XXXX

(c) How many iterations will achieve an accuracy of 10^{-3}?

(8) Define $g(x) = 1/(2 + x)$ for every $x \in [0, 1]$.

(a) Show that g has a unique fixed-point in $[0, 1]$.

(b) Use the Contraction Mapping Theorem to prove the convergence of the fixed-point iteration

$$p_{n+1} = g(p_n), \qquad p_0 = 1.$$

(c) How many iterations will achieve an accuracy of 10^{-6}?

(9) Let $f(x) = e^x + x - 3$ for every $x \in \mathbb{R}$.

(a) Show that f has a unique zero p_* that lies in $[0, 1]$.

(b) Let $p_0 = 1$. Compute the first Newton iterate p_1.

(c) Let $\{p_n\}_{n=0}^{\infty}$ be the sequence of Newton iterates with $p_0 = 1$. Why does this sequence lie above p_*.

(d) Let $p_0 = 2$ and $p_1 = 1$. Use the secant method to compute p_2.

(e) Let $\{p_n\}_{n=0}^{\infty}$ be the sequence of approximates generated by the secant method with $p_0 = 2$ and $p_1 = 1$. Why does this sequence lie above p_*.

(f) Let $p_0 = 0$ and $p_1 = 1$. Use the false-position method to compute p_2.

(g) Let $\{p_n\}_{n=0}^{\infty}$ be the sequence of approximates generated by the false-position method with $p_0 = 0$ and $p_1 = 1$. Why does this sequence lie below p_*.

(10) Consider approximating $\sqrt{5}$ by using the Newton(-Raphson) method to approximate the positive zero of $x^2 - 5 = 0$. Let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of Newton iterates corresponding to the initial guess $p_0 = 3$.

(a) Find $g(x)$ such that $p_{n+1} = g(p_n)$.

(b) Prove that $\{p_n\}_{n=0}^{\infty}$ is a decreasing sequence that lies within the interval $[\sqrt{5}, 3]$.

(c) Prove that

$$\left|p_{n+1} - \sqrt{5}\right| \leq \frac{1}{2\sqrt{5}} \left|p_n - \sqrt{5}\right|^2 .$$

(d) Use the above inequality to show that

$$\left|p_n - \sqrt{5}\right| \leq \left(\frac{1}{2\sqrt{5}}\right)^{2^n - 1} \left|3 - \sqrt{5}\right|^{2^n} .$$