Numerical Analysis II: AMSC/CMSC 667
Midterm Exam Solutions, Thursday, 14 March 2013

Give reasoning for all your answers!

(1) [35] Consider initial-value problems over $\mathbb{R}^N$ in the form

$$x' = f(t,x)\,, \qquad x(0) = x_0 \in \mathbb{R}^N\,,$$

where $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is smooth. Consider the family of one-step methods

$$x_{n+1} = x_n + (1-b)h_{n+1} f(t_n, x_n) + b h_{n+1} f(t_{n+1}, x_{n+1})\,,$$

where $h_{n+1} = t_{n+1} - t_n$ and $b \in [0,1]$ is a constant.

(a) [5] For what values of b is the method explicit?

(b) [10] For what values of b is the method second order?

(c) [10] Compute the function $g(\zeta)$ such that such that $x_n = g(h\lambda)^n x_0$ when the method is applied with a uniform step size h to

$$x' = \lambda x\,, \qquad x(0) = x_0 \in \mathbb{R}\,.$$

(d) [10] For what values of b is this method A-stable?

Solution (a). The method will be explicit if and only if its right-hand side does not depend on x_{n+1}, which will be the case if and only if $b = 0$. □

Solution (b). This method will be second order if and only if for every $x(t)$ that solves $x' = f(t,x)$ we have

$$x(t+h) = x(t) + (1-b)hf(t,x(t)) + bhf(t+h, x(t+h)) + O(h^3)$$

Because $x' = f(t,x)$, this is equivalent to showing

$$x(t+h) = x(t) + (1-b)hx'(t) + bhx'(t+h) + O(h^3)$$

Because the Taylor expansion of $x'(t+h)$ about $h = 0$ is

$$x'(t+h) = x'(t) + hx''(t) + O(h^2)\,,$$

this is equivalent to showing

$$x(t+h) = x(t) + hx'(t) + bh^2 x''(t+h) + O(h^3)$$

Because the Taylor expansion of $x(t+h)$ about $h = 0$ is

$$x(t+h) = x(t) + hx'(t) + \tfrac{1}{2}h^2 x''(t) + O(h^3)\,,$$

this is equivalent to $b = \frac{1}{2}$. Therefore this method is second order if and only if $b = \frac{1}{2}$. □

Solution (c). When this method is applied with a uniform step size h and $f(t,x) = \lambda x$, it becomes

$$x_{n+1} = x_n + (1-b)h\lambda x_n + bh\lambda x_{n+1}\,.$$

By solving this equation for x_{n+1} we see that

$$x_{n+1} = \frac{1 + (1-b)h\lambda}{1 - bh\lambda}\, x_n\,.$$

By induction it follows that

$$x_n = \left(\frac{1 + (1-b)h\lambda}{1 - bh\lambda} \right)^n x_0\,.$$

Therefore the function $g(\zeta)$ is given by

$$g(\zeta) = \frac{1 + (1-b)\zeta}{1 - b\zeta}\,.$$

□

Alternative Solution (c). When this method is applied with a uniform step size h and $f(t,x) = \lambda x$, it becomes

$$x_{n+1} = x_n + (1-b)h\lambda x_n + bh\lambda x_{n+1}\,.$$

By setting $x_n = \gamma^n x_0$ into this equation we obtain

$$\gamma^{n+1}x_0 = \gamma^n x_0 + (1-b)h\lambda\gamma^n x_0 + bh\lambda\gamma^{n+1}x_0\,.$$

When $\gamma \neq 0$ and $x_0 \neq 0$ the above equation will be satisfied if and only if

$$\gamma = 1 + (1-b)h\lambda + bh\lambda\gamma\,.$$

By solving this for γ we find that

$$\gamma = \frac{1 + (1-b)h\lambda}{1 - bh\lambda}\,.$$

Therefore the function $g(\zeta)$ is given by

$$g(\zeta) = \frac{1 + (1-b)\zeta}{1 - b\zeta}\,.$$

Remark. This alternative approach can be applied to multistep methods. □

Solution (d). A method is A-stable if $|g(\zeta)| < 1$ for every ζ in the left half-plane $\{\zeta \in \mathbb{C} : \zeta + \bar{\zeta} < 0\}$. By setting $|g(\zeta)| < 1$ we obtain

$$|1 + (1-b)\zeta|^2 < |1 - b\zeta|^2\,,$$

which is equivalent to

$$1 + (1-b)(\zeta + \bar{\zeta}) + (1-b)^2|\zeta|^2 < 1 - b(\zeta + \bar{\zeta}) + b^2|\zeta|^2\,,$$

which is equivalent to

$$(\zeta + \bar{\zeta}) + (1-2b)|\zeta|^2 < 0\,.$$

This inequality will be satisfied for every ζ in the left half-plane if and only if $b \geq \frac{1}{2}$. Therefore this method A-stable if and only if $b \geq \frac{1}{2}$. □

(2) [30] Consider initial-value problems over $\mathbb{R}^N$ in the form

$$x' = f(t,x)\,, \qquad x(0) = x_0 \in \mathbb{R}^N\,,$$

where $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is smooth. Consider the family of explicit two-step methods with uniform step size h given by

$$x_{n+1} = (1-a)x_n + ax_{n-1} + bhf(t_n, x_n) + chf(t_{n-1}, x_{n-1})\,,$$

where a, b, and c are constants.

(a) [15] For what values of a, b, and c is this method consistent to at least first-order?

(b) [15] For what values of a, b, and c will this method converge?

Solution (a). This method will be consistent to at least first order if and only if for every $x(t)$ that solves $x' = f(t,x)$ we have

$$x(t+h) = (1-a)x(t) + ax(t-h) + bhf(t,x(t)) + chf(t-h,x(t-h)) + O(h^2)\,.$$

Because $x' = f(t,x)$, this is equivalent to showing

$$x(t+h) = (1-a)x(t) + ax(t-h) + bhx'(t) + chx'(t-h) + O(h^2)\,.$$

Because the Taylor expansions of $x(t-h)$ and $x'(t-h)$ about $h=0$ are

$$x(t-h) = x(t) - hx'(t) + O(h^2)\,, \qquad x'(t-h) = x'(t) + O(h)\,,$$

this is equivalent to showing

$$x(t+h) = x(t) + (-a+b+c)hx'(t) + O(h^2)\,.$$

Because the Taylor expansion of $x(t+h)$ about $h=0$ is

$$x(t+h) = x(t) + hx'(t) + O(h^2)\,,$$

this is equivalent to $(-a+b+c) = 1$. Therefore this method is consistent to at least first order if and only if $b+c = 1+a$. □

Solution (b). This method will converge if it is consistent and stable. In part (a) we found that this method is consistent if and only if $b+c = 1+a$. Its stability is determined from the roots of the polynomial

$$p(\gamma) = \gamma^2 - (1-a)\gamma - a = (\gamma-1)(\gamma+a)\,.$$

Specifically, this method is stable if and only if every root of $p(\gamma)$ must lie in the closed unit disk $\{\gamma \in \mathbb{C} : |\gamma| \le 1\}$ and any root that lies on the unit circle $\{\gamma \in \mathbb{C} : |\gamma| = 1\}$ must be simple. Because the roots of $p(\gamma)$ are 1 and $-a$, this root condition is satisfied if and only if $a \in (-1,1]$. Therefore this method will converge if and only if

$$b+c = 1+a \qquad \text{and} \qquad a \in (-1,1]\,.$$

□

(3) [35] Consider the boundary-value problem over $[0,\ell]$

$$-u'' = x\,, \qquad u(0) = 0\,, \quad u'(\ell) + u(\ell) = 0\,.$$

- (a) [15] Give a variational formulation of this problem.
- (b) [20] Formulate a finite element method for this problem using continuous piecewise linear elements with uniform mesh spacing $h = \ell/N$. Express the result as a linear algebraic system in the form $AU = B$ where U is the N-vector of unknowns, A is an $N\times N$-matrix, and B is an N-vector. Be sure to give expressions for all the entries of A and B.

Solution (a). Solutions of $-u'' = x$ are cubic polynomials, so they are certainly in $C^\infty([0,\ell])$. Upon multiplying the equation $0 = -u'' - x$ by a test function $\tilde{v} \in C^1([0,\ell])$, integrating over $[0,\ell]$, integrating by parts once, and applying the boundary condition $u'(\ell)+u(\ell)$ yields

$$0 = \int_0^\ell \tilde{v}(-u''-x)\,\mathrm{d}x = \int_0^\ell \big(\tilde{v}'u' - \tilde{v}x\big)\,\mathrm{d}x - \tilde{v}(\ell)u'(\ell) + \tilde{v}(0)u'(0)\,.$$
$$= \int_0^\ell \big(\tilde{v}'u' - \tilde{v}x\big)\,\mathrm{d}x + \tilde{v}(\ell)u(\ell) + \tilde{v}(0)u'(0)\,.$$

Because the boundary condition at $x=0$ does not involve $u'(0)$, we must impose the boundary condition $\tilde{v}(0)=0$ on the test function $\tilde{v}$. Therefore a variational form of the boundary value problem seeks $u \in H$ such that

$$0 = \int_0^\ell \big(\tilde{v}'u' - \tilde{v}x\big)\,\mathrm{d}x + \tilde{v}(\ell)u(\ell) \qquad \text{for every } \tilde{v} \in H\,,$$

where H is the Hilbert space

$$H = \big\{ v \in H^1([0,\ell]) \,:\, v(\ell) = 0 \big\}\,,$$

and $H^1([0,\ell])$ is the usual Sobolev space over $[0,\ell]$. □

Remark. The solution u of this boundary-value problem is the minimizer over the Hilbert space H of the functional

$$F[v] = \int_0^\ell \big(\tfrac{1}{2}(v')^2 - xv\big)\,\mathrm{d}x + \tfrac{1}{2}v(\ell)^2\,.$$

Solution (b). For every $j \in \{0,1,\cdots,N\}$ set $x_j = jh$. For every $k \in \{1,2,\cdots,N\}$ let $\phi_k(x)$ be the unique function that is continuous over $[0,\ell]$, is linear over $[x_{j-1},x_j]$ for every $j \in \{1,2,\cdots,N\}$, and satisfies

$$\phi_k(x_j) = \delta_{jk} \qquad \text{for every } j \in \{0,1,\cdots,N\}\,,$$

where δ_{jk} is the Kronecker delta. The function ϕ_k is given by

$$\phi_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{h} & \text{for } x \in (x_{k-1},x_k]\,, \\ \dfrac{x_{k+1} - x}{h} & \text{for } x \in (x_k,x_{k+1}) \text{ and } k \neq N\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Its derivative is the piecewise constant function given by

$$\phi_k'(x) = \begin{cases} \dfrac{1}{h} & \text{for } x \in (x_{k-1},x_k)\,, \\ -\dfrac{1}{h} & \text{for } x \in (x_k,x_{k+1}) \text{ and } k \neq N\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Each ϕ_k is in the Hilbert space H.

Let V_h be the N-dimensional subspace of H spanned by $\{\phi_k\}_{k=1}^N$. The associated finite element method seeks $v \in V_h$ that satisfies the variational formulation

$$0 = \int_0^\ell \big(\tilde{v}'v' - \tilde{v}x\big)\,\mathrm{d}x + \tilde{v}(\ell)v(\ell) \qquad \text{for every } \tilde{v} \in V_h\,.$$

By replacing $\tilde{v}$ and v in this variational formulation with

$$\tilde{v}(x) = \sum_{k=1}^{N} \tilde{v}_k \phi_k(x)\,, \qquad v(x) = \sum_{j=1}^{N} v_j \phi_j(x)\,,$$

and invoking the arbitrariness of each $\tilde{v}_k$ leads to the system of equations

$$0 = \sum_{j=1}^{N} \int_0^\ell \phi_k'\,\phi_j'\,\mathrm{d}x\,v_j - \int_0^\ell \phi_k\,x\,\mathrm{d}x + \delta_{kN} v_N\,.$$

This system has the form $AU = B$ where the vector U, matrix A, and vector B are

$$U = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_N \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & \vdots \\ 0 & a_{32} & a_{33} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{(N-1)N} \\ 0 & \cdots & 0 & a_{N(N-1)} & a_{NN} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_N \end{pmatrix},$$

with

$$a_{kj} = \int_0^\ell \phi_k'\,\phi_j'\,\mathrm{d}x + \delta_{kN}\delta_{Nj}\,, \qquad b_k = \int_0^\ell \phi_k\,x\,\mathrm{d}x\,.$$

Finally, these integrals can be evaluated to find

$$\begin{aligned}
a_{kk} &= \int_{x_{k-1}}^{x_{k+1}} (\phi_k')^2\,\mathrm{d}x = \int_{x_{k-1}}^{x_{k+1}} \frac{1}{h^2}\,\mathrm{d}x = \frac{2}{h} \qquad \text{for } k \in \{1, 2, \cdots, N-1\}\,,\\
a_{NN} &= \int_{x_{N-1}}^{x_N} (\phi_N')^2\,\mathrm{d}x + 1 = \int_{x_{N-1}}^{x_N} \frac{1}{h^2}\,\mathrm{d}x + 1 = \frac{1}{h} + 1\,,\\
a_{k(k-1)} = a_{(k-1)k} &= \int_{x_{k-1}}^{x_k} \phi_{k-1}'\phi_k'\,\mathrm{d}x = -\int_{x_{k-1}}^{x_k} \frac{1}{h^2}\,\mathrm{d}x = -\frac{1}{h} \qquad \text{for } k \in \{2, 3 \cdots, N\}\,,\\
b_k &= \int_{x_{k-1}}^{x_{k+1}} \phi_k\,x\,\mathrm{d}x = x_k \int_{x_{k-1}}^{x_{k+1}} \phi_k\,\mathrm{d}x + \int_{x_{k-1}}^{x_{k+1}} \phi_k\,(x - x_k)\,\mathrm{d}x\\
&= x_k h \qquad \text{for } k \in \{1, 2, \cdots, N-1\}\,,\\
b_N &= \int_{x_{N-1}}^{x_N} \phi_N\,x\,\mathrm{d}x = x_N \int_{x_{N-1}}^{x_N} \phi_N \mathrm{d}x + \int_{x_{N-1}}^{x_N} \phi_k\,(x - x_N)\,\mathrm{d}x\\
&= x_N \int_{x_{N-1}}^{x_N} \frac{x - x_{N-1}}{h}\,\mathrm{d}x - \int_{x_{N-1}}^{x_N} \frac{(x - x_{N-1})(x_N - x)}{h}\,\mathrm{d}x\\
&= x_N \tfrac{1}{2} h - h^2 \int_0^h s(1-s)\,\mathrm{d}s = \tfrac{1}{2} x_N h - \tfrac{1}{6} h^2\,,
\end{aligned}$$

where $h = \ell/N$. When computing b_k for $k \in \{1, 2, \cdots, N-1\}$ above we have used the fact that the integrand has odd symmetry about $x = x_k$ to see that

$$\int_{x_{k-1}}^{x_{k+1}} \phi_k\,(x - x_k)\,\mathrm{d}x = 0\,.$$

All of the integrals above except the last one in the computation of b_N can be evaluated by the area formulas for triangles or rectangles, or by odd symmetry. □