Quiz 12 Solutions, Math 246, Professor David Levermore
Tuesday, 11 December 2012

(1) [5] Consider the first-order planar system

$$x' = 2x - y\,, \qquad y' = -2y + x^2\,.$$

(a) Find all of its stationary points.
(b) Find a nonconstant function $H(x, y)$ such that every trajectory of this system satisfies $H(x, y) = c$ for some constant c.

Solution (a). The stationary points satisfy

$$0 = 2x - y\,, \qquad 0 = -2y + x^2\,.$$

The first equation is satisfied if and only if $y = 2x$, whereby the second becomes $0 = -4x + x^2$, which is solved by $x = 0$ or $x = 4$. All the stationary points are therefore $(0, 0)$ and $(4, 8)$.

Solution (b). The system is Hamiltonian because

$$\partial_x(2x - y) + \partial_y(-2y + x^2) = 2 - 2 = 0\,.$$

Therefore there exists $H(x, y)$ such that

$$\partial_y H(x, y) = 2x - y\,, \qquad -\partial_x H(x, y) = -2y + x^2\,.$$

By integrating the first equation we find $H(x, y) = 2xy - \frac{1}{2}y^2 + h(x)$. By substituting this into the second equation we see that $2y + h'(x) = 2y - x^2$, whereby $h'(x) = -x^2$. Therefore we can set $H(x, y) = 2xy - \frac{1}{2}y^2 - \frac{1}{3}x^3$.

(2) [5] Consider the first-order planar system

$$x' = y\,, \qquad y' = 5x - x^2 + 4y\,.$$

Its stationary points are $(0, 0)$ and $(5, 0)$. Classify the type and stability of each of these stationary points. (You do not have to sketch anything.)

Solution. The Jacobian matrix is $\mathbf{J}(x, y) = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 5 - 2x & 4 \end{pmatrix}$.

- At $(0, 0)$ the coefficient matrix of its linearization is $\mathbf{A} = \mathbf{J}(0, 0) = \begin{pmatrix} 0 & 1 \\ 5 & 4 \end{pmatrix}$, which has characteristic polynomial $p(z) = z^2 - 4z - 5 = (z + 1)(z - 5)$. The eigenvalues of $\mathbf{A}$ are -1 and 5. Because these are real, nonzero, and have opposite sign, the stationary point $(0, 0)$ is a *saddle* and thereby is *unstable*.
- At $(5, 0)$ the coefficient matrix of its linearization is $\mathbf{A} = \mathbf{J}(5, 0) = \begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix}$, which has characteristic polynomial $p(z) = z^2 - 4z + 5 = (z - 2)^2 + 1$. The eigenvalues of $\mathbf{A}$ are the conjugate pair $2 \pm i$. Because these have positive real part, and because $a_{21} = -5 < 0$, the stationary point $(5, 0)$ is a *clockwise spiral source* and thereby is *repelling*.