Quiz 11 Solutions, Math 246, Professor David Levermore
Tuesday, 27 November 2012

(1) [2] Find the eigenvalues of $\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix}$.

Solution. Because $\mathbf{A}$ is 2×2 its characteristic polynomial is

$$\begin{aligned} p(z) &= z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - (2-3)z + (2 \cdot (-3) - (-2) \cdot 2) = z^2 + z - 2 = (z-1)(z+2) \,. \end{aligned}$$

Therefore the eigenvalues of $\mathbf{A}$ are 1 and -2.

(2) [3] $\mathbf{A} = \begin{pmatrix} 3 & 7 \\ 2 & -2 \end{pmatrix}$ has eigenvalues 5 and -4. Find an eigenvector for each eigenvalue.

Solution. We have $\mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -2 & 7 \\ 2 & -7 \end{pmatrix}$ and $\mathbf{A} + 4\mathbf{I} = \begin{pmatrix} 7 & 7 \\ 2 & 2 \end{pmatrix}$.
The eigenvectors $\mathbf{v}_1$ associated with the eigenvalue 5 satisfy $(\mathbf{A} - 5\mathbf{I})\mathbf{v}_1 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} + 4\mathbf{I}$ that

$$\mathbf{v}_1 = \alpha_1 \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0 \,.$$

The eigenvectors $\mathbf{v}_2$ associated with the eigenvalue -4 satisfy $(\mathbf{A} + 4\mathbf{I})\mathbf{v}_2 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} - 5\mathbf{I}$ that

$$\mathbf{v}_2 = \alpha_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0 \,.$$

(3) [2] A real 2×2 matrix $\mathbf{A}$ has the eigenpairs $\left(3 \,, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$ and $\left(-1 \,, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$. Find a fundamental matrix $\mathbf{\Psi}(t)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution. A fundamental matrix is

$$\mathbf{\Psi}(t) = \left(e^{3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \;\; e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 2e^{3t} & e^{-t} \\ -e^{3t} & 2e^{-t} \end{pmatrix} \,.$$

(4) [3] A real 2×2 matrix $\mathbf{A}$ has the eigenpair $\left(5 + i3 \,, \begin{pmatrix} 2 \\ -i \end{pmatrix}\right)$. Diagonalize $\mathbf{A}$.

Solution. Because $\mathbf{A}$ is real, a second eigenpair will be the complex conjugate of the given eigenpair. If we use the eigenpairs

$$\left(5 + i3 \,, \begin{pmatrix} 2 \\ -i \end{pmatrix}\right) \,, \qquad \left(5 - i3 \,, \begin{pmatrix} 2 \\ i \end{pmatrix}\right) \,,$$

then set

$$\mathbf{V} = \begin{pmatrix} 2 & 2 \\ -i & i \end{pmatrix} \,, \qquad \mathbf{D} = \begin{pmatrix} 5 + i3 & 0 \\ 0 & 5 - i3 \end{pmatrix} \,.$$

Because $\det(\mathbf{V}) = 2 \cdot i - (-i) \cdot 2 = i4$, we obtain the diagonalization

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 2 & 2 \\ -i & i \end{pmatrix} \begin{pmatrix} 5 + i3 & 0 \\ 0 & 5 - i3 \end{pmatrix} \frac{1}{i4} \begin{pmatrix} i & -2 \\ i & 2 \end{pmatrix} \,.$$