Quiz 10 Solutions, Math 246, Professor David Levermore
Tuesday, 20 November 2012

(1) [4] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 3+t^5 \\ t^3 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 3+t^5 & t^2 \\ t^3 & 1 \end{pmatrix} = (3+t^5)\cdot 1 - t^2 \cdot t^3 = 3\,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} 3+t^5 & t^2 \\ t^3 & 1 \end{pmatrix}$. Because $\dfrac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, we have

$$\begin{aligned}
\mathbf{A}(t) &= \frac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t}\,\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 5t^4 & 2t \\ 3t^2 & 0 \end{pmatrix} \begin{pmatrix} 3+t^5 & t^2 \\ t^3 & 1 \end{pmatrix}^{-1} \\
&= \frac{1}{3}\begin{pmatrix} 5t^4 & 2t \\ 3t^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t^2 \\ -t^3 & 3+t^5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3t^4 & 6t-3t^6 \\ 3t^2 & -3t^4 \end{pmatrix} = \begin{pmatrix} t^4 & 2t-t^6 \\ t^2 & -t^4 \end{pmatrix}\,.
\end{aligned}$$

(2) [2] Suppose $e^{t\mathbf{A}} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$. What is the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$?

Solution. A general solution is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}\,.$$

(3) [4] Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$. Compute $e^{t\mathbf{A}}$.

Solution. The characteristic polynomial is $p(z) = z^2 - 6z + 13 = (z-3)^2 + 2^2$. Hence,

$$\begin{aligned}
e^{t\mathbf{A}} &= e^{3t}\left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A} - 3\mathbf{I})\right] \\
&= e^{3t}\left[\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\right] = e^{3t}\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}\,.
\end{aligned}$$