Quiz 9 Solutions, Math 246, Professor David Levermore
Tuesday, 13 November 2012

Short Table: $\mathcal{L}[e^{at}](s) = \dfrac{1}{s-a}$ for $s > a$, $\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$.

(1) [3] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem
$y'' + 9y = 0$, $y(0) = 0$, $y'(0) = 2$. DO NOT solve for $y(t)$, just $Y(s)$!

Solution. The Laplace transform of the initial-value problem gives

$$\mathcal{L}[y''](s) + 9\mathcal{L}[y](s) = 0\,,$$

where

$$\begin{aligned}
\mathcal{L}[y](s) &= Y(s)\,,\\
\mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s)\,,\\
\mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2\,.
\end{aligned}$$

Hence,

$$(s^2+9)Y(s) = 2\,, \qquad \Longrightarrow \qquad Y(s) = \frac{2}{s^2+9}\,.$$

(2) [3] Find the inverse Laplace transform $y(t)$ of the function $Y(s) = \dfrac{2s+3}{s^2+3s-10}$.

Solution. We have the partial fraction identity

$$Y(s) = \frac{2s+3}{s^2+3s-10} = \frac{2s+3}{(s-2)(s+5)} = \frac{1}{s-2} + \frac{1}{s+5}\,.$$

Item 1 in the table at the top of the page with $a = 2$ and with $a = -5$ then gives

$$y(t) = \mathcal{L}^{-1}[Y](t) = \mathcal{L}^{-1}\left[\frac{1}{s-2} + \frac{1}{s+5}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+5}\right] = e^{2t} + e^{-5t}\,.$$

(3) [3] Transform the equation $v''' + (v'')^2 - v^3v' - e^tv^4 = t^2$ into a first-order system of ordinary differential equations.

Solution. Because the equation is third order, the first-order system must have dimension three. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ t^2 - x_3^2 + x_1^3x_2 + e^tx_1^4 \end{pmatrix}\,, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} v \\ v' \\ v'' \end{pmatrix}\,.$$

(4) [1] Compute $\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution.

$$\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 + (-2)\cdot 2 \\ (-2)\cdot 1 + 1\cdot 2 \end{pmatrix} = \begin{pmatrix} 3-4 \\ -2+2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}\,.$$