Quiz 8, Math 246, Professor David Levermore
Tuesday, 6 November 2012

(1) [2] Sketch a graph of the function $u(t-5)$ over $0 \leq t \leq 10$.

Solution. Because

$$u(t-5) = \begin{cases} 0 & \text{for } 0 \leq t < 5\,, \\ 1 & \text{for } 5 \leq t \leq \infty\,, \end{cases}$$

the graph of $u(t-5)$ over $0 \leq t \leq 10$ is indicated below by the plus signs.

```
                              • + + + + + + + + + + + + + + +

| + + + + + + + + + + + + + + ∘ − − − − − − − − − − − − − |
0                             5                            10
```

(2) [3] Give the exponential order α as $t \to \infty$ of the following functions.

(a) e^{7t} **Solution.** $\alpha = 7$

(b) $t^2 \sin(3t)$ **Solution.** $\alpha = 0$

(c) $t^3 e^{-2t}$ **Solution.** $\alpha = -2$

(3) [5] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for the function $f(t) = e^{-3t}u(t-5)$, where u is the unit step function.

Solution. By the definitions of the Laplace transform and the unit step function

$$\begin{aligned} \mathcal{L}[f](s) &= \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{-st} e^{-3t} u(t-5)\,\mathrm{d}t \\ &= \lim_{T\to\infty} \int_5^T e^{-st} e^{-3t}\,\mathrm{d}t = \lim_{T\to\infty} \int_5^T e^{-(s+3)t}\,\mathrm{d}t\,. \end{aligned}$$

For $s \leq -3$ the above limit diverges because $e^{-(s+3)t} \geq 1$. For $s > -3$

$$\int_5^T e^{-(s+3)t}\,\mathrm{d}t = -\frac{e^{-(s+3)t}}{s+3}\bigg|_5^T = \frac{e^{-(s+3)5}}{s+3} - \frac{e^{-(s+3)T}}{s+3}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \left[\frac{e^{-(s+3)5}}{s+3} - \frac{e^{-(s+3)T}}{s+3} \right] = \frac{e^{-(s+3)5}}{s+3} \quad \text{for } s > -3\,.$$