

Quiz 7 Solutions, Math 246, Professor David Levermore
Tuesday, 23 October 2012

- (1) [3] Find the Green function $g(t)$ associated with the differential operator $L = D^2 + 9$.

Solution. The Green function $g(t)$ satisfies the initial-value problem

$$g'' + 9g = 0, \quad g(0) = 0, \quad g'(0) = 1.$$

The characteristic polynomial is $p(z) = z^2 + 3^2$, which has the conjugate roots $\pm i3$. Therefore $g(t)$ has the form

$$g(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

Because $g'(t) = -3c_1 \sin(3t) + 3c_2 \cos(3t)$, the initial conditions imply

$$0 = g(0) = c_1 \cos(0) + c_2 \sin(0) = c_1,$$

$$1 = g'(0) = -3c_1 \sin(0) + 3c_2 \cos(0) = 3c_2.$$

The solution of this system is $c_1 = 0$, $c_2 = \frac{1}{3}$. Therefore the Green function $g(t)$ is

$$g(t) = \frac{1}{3} \sin(3t).$$

- (2) [3] Give a particular solution of the equation $y'' - 4y = 4e^{-2t}$.

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2 - 4 = (z + 2)(z - 2)$, which has roots -2 and 2 . Its forcing has degree $d = 0$, characteristic $\mu + i\nu = -2$, and multiplicity $m = 1$.

Key Identity Evaluation. Because $m = 1$ and $d + m = 1$, we need the first derivative of the Key Identity, which we compute as

$$L(e^{zt}) = (z^2 - 4)e^{zt},$$

$$L(te^{zt}) = (z^2 - 4)te^{zt} + 2ze^{zt}.$$

By evaluating the derivative at the forcing characteristic $z = -2$ we see that

$$L(te^{-2t}) = ((-2)^2 - 4)te^{-2t} + 2(-2)e^{-2t} = -4e^{-2t}.$$

Therefore a particular solution is $y_P(t) = -te^{-2t}$.

Undetermined Coefficients. Because $d = 0$, $\mu + i\nu = -2$ and $m = 1$, there is a particular solution of the form

$$y_P(t) = At e^{-2t}.$$

Because

$$y_P'(t) = -2At e^{-2t} + Ae^{-2t}, \quad y_P''(t) = 4At e^{-2t} - 4Ae^{-2t},$$

we find that

$$\begin{aligned} y_P'' - 4y_P &= (4At e^{-2t} - 4Ae^{-2t}) - 4At e^{-2t} \\ &= -4Ae^{-2t}. \end{aligned}$$

By setting $-4A = 4$ we find $A = -1$ and the particular solution $y_P(t) = -te^{-2t}$.

Green Function. To apply the Green method, first we must show (as in problem 1) that the Green function is given by $g(t) = \frac{1}{4}(e^{2t} - e^{-2t})$. Then a particular solution is given by

$$\begin{aligned} y_p(t) &= \frac{1}{4} \int_0^t (e^{2t-2s} - e^{2s-2t}) \cdot 4e^{-2s} ds \\ &= e^{2t} \int_0^t e^{-4s} ds - e^{-2t} \int_0^t ds = e^{2t} \frac{e^{-4s}}{-4} \Big|_0^t - e^{-2t}t \\ &= e^{2t} \left(\frac{e^{-4t}}{-4} - \frac{1}{-4} \right) - e^{-2t}t = \frac{e^{2t} - e^{-2t}}{4} - e^{-2t}t. \end{aligned}$$

This is the unique particular solution that satisfies $y_p(0) = 0$ and $y'_p(0) = 0$. It is a different particular solution than that obtained by either Key Identity Evaluations or Undetermined Coefficients.

Remark. The Green Function method generally takes much longer than either the Key Identity Evaluations or Undetermined Coefficients methods! In the above problem the integrals are relatively simple, so the difference is less dramatic than in most problems, but even in this case it takes longer.

- (3) [4] A spring-mass system is governed by the initial-value problem

$$h'' + \mu h' + 25h = 0, \quad h(0) = 3, \quad h'(0) = -2.$$

- (a) Determine the natural frequency and period of the spring.
 (b) For what value of μ is the system critically damped?

Solution (a). Because the equation is in normal form, the natural frequency ω_o is

$$\omega_o = \sqrt{25} = 5.$$

The natural period is therefore $T_o = 2\pi/\omega_o = \frac{2}{5}\pi$.

Solution (b). The characteristic polynomial is

$$p(z) = z^2 + \mu z + 25 = (z + \frac{1}{2}\mu)^2 + 25 - \frac{1}{4}\mu^2.$$

The system will be critically damped when $25 = \frac{1}{4}\mu^2$, which is when

$$\mu = \sqrt{4 \cdot 25} = \sqrt{100} = 10.$$

Remark. Key Identity Evaluations can be used to solve the third problem on last week's quiz as follows — namely, give a particular solution of the equation

$$y'' + 4y' + 13y = 5e^{-t}.$$

Solution. Recall that this forcing has degree $d = 0$, characteristic $\mu + i\nu = -1$, and multiplicity $m = 0$.

Key Identity Evaluation. Because $d + m = 0$, we just need the Key Identity, which is

$$L(e^{zt}) = (z^2 + 4z + 13)e^{zt}.$$

By evaluating this at the forcing characteristic $z = -1$ we see that

$$L(e^{-t}) = ((-1)^2 + 4(-1) + 13)e^{-t} = (1 - 4 + 13)e^{-t} = 10e^{-t}.$$

Therefore a particular solution is $y_p(t) = \frac{1}{2}e^{-t}$.