Quiz 6 Solutions, Math 246, Professor David Levermore
Tuesday, 16 October 2012

(1) [5] Give a general solution of the equation

$$(\mathrm{D}+7)^3(\mathrm{D}^2-6\mathrm{D}+13)^2 y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a seventh-order, homogeneous, linear equation with constant coefficients. Its characteristic polynomial is

$$p(z) = (z+7)^3(z^2-6z+13)^2 = (z+7)^3\big((z-3)^2+2^2\big)^2\,,$$

which has roots -7, -7, -7, $3+i2$, $3+i2$, $3-i2$, and $3-i2$. A general solution of the differential equation is

$$\begin{aligned} y(t) = c_1 e^{-7t} + c_2 t\, e^{-7t} + c_3 t^2 e^{-7t} + c_4 e^{3t}\cos(2t) + c_5 e^{3t}\sin(2t) \\ + c_6 t\, e^{3t}\cos(2t) + c_7 t\, e^{3t}\sin(2t)\,. \end{aligned}$$

The reasoning is as follows:

- the triple real root -7 yields the solutions e^{-7t}, $t\,e^{-7t}$, and t^2e^{-7t};
- the double conjugate pair $3 \pm i2$ yields the solutions

$$e^{3t}\cos(2t)\,, \quad e^{3t}\sin(2t)\,, \quad t\,e^{3t}\cos(2t)\,, \quad \text{and} \quad t\,e^{3t}\sin(2t)\,.$$

(2) [3] Give the degree, characteristic, and multiplicity for the forcing term of the equation

$$y'' + 4y' + 13y = 3t^4 e^{-2t}\sin(3t)\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2+4z+13 = (z+2)^2+3^2$, which has roots $-2 \pm i3$.

The forcing term $3t^4e^{-2t}\sin(3t)$ has degree $d=4$, characteristic $\mu + i\nu = -2+i3$, and multiplicity $m=1$.

(3) [2] Give a particular solution of the equation

$$y'' + 4y' + 13y = 5e^{-t}\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2+4z+13 = (z+2)^2+3^2$, which has roots $-2 \pm i3$. Its forcing has degree $d=0$, characteristic $\mu+i\nu=-1$, and multiplicity $m=0$.

Undetermined Coefficients. Because $\mu + i\nu = -1$ and $d = m = 0$, there is a particular solution of the form $y_P(t) = Ae^{-t}$. Because

$$y_P'(t) = -Ae^{-t}\,, \qquad y_P''(t) = Ae^{-t}\,,$$

we find that

$$\begin{aligned} y_P'' + 4y_P' + 13y_P &= Ae^{-t} + 4\cdot(-Ae^{-t}) + 13Ae^{-t} \\ &= (1-4+13)Ae^{-t} = 10Ae^{-t} \\ &= 5e^{-t}\,. \end{aligned}$$

By setting $10A = 5$ we obtain $A = \frac{1}{2}$, which gives the particular solution $y_P(t) = \frac{1}{2}e^{-t}$.