Quiz 5 Solutions, Math 246, Professor David Levermore
Tuesday, 9 October 2012

(1) [2] What is the interval of definition for the solution to the initial-value problem

$$z''' - \frac{e^t}{t}\, z' + \frac{\sin(3t)}{5-t}\, z = \frac{3}{4+t}\,, \qquad z(1) = z'(1) = z''(1) = 4\,.$$

Solution. This linear equation is already in normal form. Both of its coefficients are defined and continuous everywhere except $t = 0$ and $t = 5$. Its forcing is defined and continuous everywhere except $t = -4$. The initial time is $t = 1$. The interval of definition is therefore $(0, 5)$.

(2) [3] Compute the Wronskian $W[Y_1, Y_2](t)$ of the functions $Y_1(t) = 1+t$ and $Y_2(t) = e^t$. (Evaluate the determinant and simplify.)

Solution. Because $Y_1'(t) = 1$ and $Y_2'(t) = e^t$, the Wronskian is

$$\begin{aligned} W[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} 1+t & e^t \\ 1 & e^t \end{pmatrix} \\ &= (1+t)e^t - e^t = te^t\,. \end{aligned}$$

(3) [1] Suppose that $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$ are solutions of the differential equation

$$y''' + b(t)y' + c(t)y = 0\,,$$

where $b(t)$ and $c(t)$ are continuous over $(-8, 8)$. Suppose that $W[Y_1, Y_2, Y_3](0) = 3$. What is $W[Y_1, Y_2, Y_3](4)$?

Solution. The equation is in normal form and has coefficients that are continuous over $(-8, 8)$. Because it is third-order while the coefficient of y'' is zero, the Abel Theorem implies that $W[Y_1, Y_2, Y_3](t)$ is constant. We can thereby conclude that $W[Y_1, Y_2, Y_3](4) = W[Y_1, Y_2, Y_3](0) = 3$.

(4) [4] Given that $\cos(4t)$ and $\sin(4t)$ are linearly independent solutions of $y'' + 16y = 0$, find the natural fundamental set of solutions of this equations associated with $t = 0$.

Solution. The general initial-value problem associated with $t = 0$ is

$$y'' + 16y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

Because $\cos(4t)$ and $\sin(4t)$ are linearly independent solutions, a general solution is $Y(t) = c_1\cos(4t) + c_2\sin(4t)$. Then $Y'(t) = -4c_1\sin(4t) + 4c_2\cos(4t)$ and the initial conditions yield

$$y_0 = Y(0) = c_1\,, \qquad\qquad y_1 = Y'(0) = 4c_2\,.$$

It follows that $c_1 = y_0$ and $c_2 = y_1/4$. Hence, the solution of the general initial-value problem is

$$Y(t) = y_0\cos(4t) + y_1\frac{\sin(4t)}{4}\,.$$

Therefore the natural fundamental set of solutions associated with $t = 0$ is

$$N_0(t) = \cos(4t)\,, \qquad N_1(t) = \frac{\sin(4t)}{4}\,.$$