Quiz 4 Solutions, Math 246, Professor David Levermore
Tuesday, 25 September 2012

(1) [4] Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = ti; y(1) = yi; h = (tf - ti)/n;
for j = 1:n
z = t(j)^2 − y(j)^3;
t(j + 1) = t(j) + h;
y(j + 1) = y(j) + (h/2)*(z + t(j + 1)^2 − (y(j) + h*z)^3);
end
```

(a) What initial-value problem is being approximated numerically?
(b) Identify the numerical method being used and give its order.

Solution. The initial-value problem being approximated is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 - y^3\,, \qquad y(\mathrm{ti}) = \mathrm{yi}\,.$$

The Runge-trapezoidal (improved Euler) method is being used, which is second order.

(2) [1] Suppose you are using the Runge-midpoint method to numerically approximate the solution of an initial-value problem over the time interval $[0, 5]$. By what factor would you expect the global error to decrease when the number of time steps that you take is increased from 500 to 2000.

Solution. When the number of time steps is increased by a factor of 4, the time step h is reduced by a factor of $1/4$. Because the Runge-midpoint method is second order, the error is thereby reduced by a factor of $(1/4)^2 = 1/16$.

(3) [5] Find an implicit general solution of the differential form

$$(6x^2 + 2xy)\,\mathrm{d}x + (x^2 + 4y^3)\,\mathrm{d}y = 0\,.$$

Solution. This differential form is exact because

$$\partial_y(6x^2 + 2xy) = 2x \qquad = \qquad \partial_x(x^2 + 4y^3) = 2x\,.$$

Therefore we can find $H(x, y)$ such that

$$\partial_x H(x, y) = 6x^2 + 2xy\,, \qquad \partial_y H(x, y) = x^2 + 4y^3\,.$$

Upon integrating the first equation with respect to x we find

$$H(x, y) = \int 6x^2 + 2xy\,\mathrm{d}x = 2x^3 + x^2y + h(y)\,.$$

This implies that $\partial_y H(x, y) = x^2 + h'(y)$. Plugging this into the left-hand side of the second equation gives $x^2 + h'(y) = x^2 + 4y^3$, which yields $h'(y) = 4y^3$. Taking $h(y) = y^4$ gives $H(x, y) = 2x^3 + x^2y + y^4$. Hence, an implicit general solution is

$$2x^3 + x^2y + y^4 = c\,.$$

Remark. This cannot easily be solved for y to obtain an explicit general solution.