Quiz 2 Solutions, Math 246, Professor David Levermore
Tuesday, 11 September 2012

(1) [5] Find the solution of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t^2 e^{-3x}\,, \qquad x(0) = x_I \quad \text{for some } x_I \text{ in } (-\infty, \infty)\,.$$

Give its interval of definition in terms of x_I.

Solution. This equation is separable. Its separated differential form is $e^{3x}\,\mathrm{d}x = t^3\,\mathrm{d}t$, whereby

$$\int e^{3x}\,\mathrm{d}x = \int t^2\,dt\,.$$

By integrating both sides gives $\frac{1}{3}e^{3x} = \frac{1}{3}t^3 + c$. The initial condition then implies that $\frac{1}{3}e^{3x_I} = \frac{1}{3}0^3 + c$, whereby $c = \frac{1}{3}e^{3x_I}$. The solution is thereby given implicity by

$$e^{3x} = t^3 + e^{3x_I}\,.$$

Whenever $t^3 + e^{3x_I} > 0$ this can be solved explicitly for x to obtain

$$x = \tfrac{1}{3}\log\bigl(t^3 + e^{3x_I}\bigr)\,.$$

The condition $t^3 + e^{3x_I} > 0$ implies that $t^3 > -e^{3x_I}$, which implies that $t > -e^{x_I}$. Therefore the interval of definition is $(-e^{x_I}, \infty)$.

(2) [5] Sketch the phase-line portrait for the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{(u+2)^2(u-2)(u-6)}{4+u^2}\,.$$

Identify every stationary (equilibrium) point as either stable, unstable, or semistable. (You do not have to find the solution!)

Solution. The stationary points are found by setting

$$(u+2)^2(u-2)(u-6) = 0\,.$$

Therefore the stationary points are $u = -2$, $u = 2$, and $u = 6$. The phase-line portrait is

$$\begin{array}{ccccccccc} + & & + & & - & & + & & \\ \to\to\to\to & \bullet & \to\to\to\to & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \to\to\to\to & & u \\ & -2 & & 2 & & 6 & & & \\ & \text{semistable} & & \text{stable} & & \text{unstable} & & & \end{array}$$

Remark. If $u(0)$ lies within the interval $[-2, 6]$ then the interval of definition of the solution $u(t)$ is $(-\infty, \infty)$. Do you see why?