Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 4 September 2012

(1) [4] For each of the following ordinary differential equations, give its order and state whether it is linear or nonlinear.

(a) $\dfrac{d^3y}{dz^3} - z^2\dfrac{dy}{dz} = z^4 - e^z y;$ **Solution.** third order, linear.

(b) $\dfrac{d^2x}{dt^2} = x\,\dfrac{dx}{dt} + \sin(t).$ **Solution.** second order, nonlinear.

(2) [4] Solve the initial-value problem

$$t\frac{dz}{dt} = 3z + t^4\,, \qquad z(1) = 3\,.$$

Solution. This equation is linear. Its normal form is

$$\frac{dz}{dt} - \frac{3}{t}\,z = t^3\,.$$

An integrating factor is $e^{A(t)}$ where $A'(t) = -3/t$. Setting $A(t) = -3\log(t)$, we find that $e^{A(t)} = e^{-3\log(t)} = t^{-3}$. Hence, the problem has the intgrating factor form

$$\frac{d}{dt}\big(t^{-3}z\big) = t^{-3}\cdot t^3 = 1\,.$$

Integrating both sides yields

$$t^{-3}z = t + c\,.$$

Imposing the initial condition gives

$$1^{-2}\cdot 3 = 1 + c\,,$$

whereby $c = 2$. Therefore the solution is

$$z = t^3(t+2) = t^4 + 2t^3\,, \qquad \text{for every } t > 0\,.$$

Remark. The interval of definition for this solution is $(0,\infty)$. Can you see why?

(3) [2] Give the interval of definition for the solution of the initial-value problem

$$\frac{dz}{dt} + \frac{z}{\sin(t)} = \frac{1}{t^2 - 25}\,, \qquad z(4) = 2\,.$$

(You do not have to solve this equation to answer this question!)

Solution. This equation is linear and is already in normal form. The coefficient $1/\sin(t)$ is continuous everywhere except where $t = n\pi$ for some integer n, while the forcing $1/(t^2 - 25)$ is continuous everywhere except at $t = \pm 5$. Therefore you can read off that the interval of definition for its solution is $(\pi, 5)$ because:

- the initial time $t = 4$ is in $(\pi, 5)$,
- the coefficient and forcing are both continuous over $(\pi, 5)$,
- the coefficient is not defined at $t = \pi$,
- the forcing is not defined at $t = 5$.