HIGHER-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS IV: Laplace Transform Method

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Because the presentation of this material in lecture will differ from that in the book, I felt that notes that closely follow the lecture presentation might be appreciated.

Contents

8. Laplace Transform Method

The Laplace transform allows us to transform an initial-value problem for a linear ordinary differential equation *with constant coefficients* into a linear algebaric equation that can be easily solved. The solution of the initial-value problem can be obtained from the solution of the algebaric equation by taking the so-called inverse Laplace transform.

8.1. **Definition of the Transform.**

The Laplace transform of a function $f(t)$ defined over $t \geq 0$ is another function $\mathcal{L}[f](s)$ that is formally defined by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,. \tag{8.1}$$

You should recall from calculus that the above definite integral is improper because its upper endpoint is ∞. Because improper definite integrals are defined by limits, the correct definition of the Laplace transform is

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t\,, \tag{8.2}$$

provided that the definite integrals over $[0, T]$ appearing in the above limit are proper. The Laplace transform $\mathcal{L}[f](s)$ is defined only at those s for which this limit exists.

Example. Use definition (8.2) to compute $\mathcal{L}[e^{at}](s)$ for any real a.

Solution. From (8.2) we see that for any $s \neq a$ we have

$$\begin{aligned}
\mathcal{L}[e^{at}](s) &= \lim_{T\to\infty} \int_0^T e^{-st} e^{at}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{(a-s)t}\,\mathrm{d}t \\
&= \lim_{T\to\infty} \frac{e^{(a-s)t}}{a-s}\bigg|_{t=0}^{T} = \lim_{T\to\infty} \left[\frac{1}{s-a} - \frac{e^{(a-s)T}}{s-a}\right] = \begin{cases} \dfrac{1}{s-a} & \text{for } s > a\,, \\ \infty & \text{for } s < a\,, \end{cases}
\end{aligned}$$

while for $s = a$ we have

$$\mathcal{L}[e^{at}](s) = \lim_{T\to\infty} \int_0^T e^{-(s-a)t}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T \mathrm{d}t = \lim_{T\to\infty} T = \infty\,.$$

Therefore $\mathcal{L}[e^{at}](s)$ is only defined for $s > a$ with

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a} \quad \text{for } s > a\,.$$

Example. Use definition (8.2) to compute $\mathcal{L}[t\,e^{at}](s)$ for any real a.

Solution. From (8.2) we see that for any $s \neq a$ we have

$$\begin{aligned}
\mathcal{L}[t\,e^{at}](s) &= \lim_{T\to\infty} \int_0^T t\,e^{-st}e^{at}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T t\,e^{(a-s)t}\,\mathrm{d}t \\
&= \lim_{T\to\infty} \left(\frac{t}{a-s} - \frac{1}{(a-s)^2} \right) e^{(a-s)t} \bigg|_{t=0}^{T} \\
&= \lim_{T\to\infty} \left[\frac{1}{(s-a)^2} - \left(\frac{T}{s-a} + \frac{1}{(s-a)^2} \right) e^{(a-s)T} \right] = \begin{cases} \dfrac{1}{(s-a)^2} & \text{for } s > a\,, \\ \infty & \text{for } s < a\,, \end{cases}
\end{aligned}$$

while for $s = a$ we have

$$\mathcal{L}[t\,e^{at}](s) = \lim_{T\to\infty} \int_0^T t\,e^{-(s-a)t}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T t\,\mathrm{d}t = \lim_{T\to\infty} \tfrac{1}{2}T^2 = \infty\,.$$

Therefore $\mathcal{L}[t\,e^{at}](s)$ is only defined for $s > a$ with

$$\mathcal{L}[t\,e^{at}](s) = \frac{1}{(s-a)^2} \quad \text{for } s > a\,.$$

Example. Use definition (8.2) to compute $\mathcal{L}[e^{ibt}](s)$ for any real b.

Solution. For $b \neq 0$ we see from (8.2) that for any real s we have

$$\begin{aligned}
\mathcal{L}[e^{ibt}](s) &= \lim_{T\to\infty} \int_0^T e^{-st}e^{ibt}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{-(s-ib)t}\,\mathrm{d}t = \lim_{T\to\infty} \left(-\frac{e^{-(s-ib)t}}{s-ib} \right) \bigg|_{t=0}^{T} \\
&= \lim_{T\to\infty} \left[\frac{1}{s-ib} - \frac{e^{-(s-ib)T}}{s-ib} \right] = \begin{cases} \dfrac{1}{s-ib} & \text{for } s > 0\,, \\ \text{undefined} & \text{for } s \leq 0\,. \end{cases}
\end{aligned}$$

The case $b = 0$ is identical to our first example with $a = 0$. In every case $\mathcal{L}[e^{ibt}](s)$ is only defined for $s > 0$ with

$$\mathcal{L}[e^{ibt}](s) = \frac{1}{s-ib} \quad \text{for } s > 0\,.$$

8.2. **Properties of the Transform.** If we always had to return to the definition of the Laplace transform everytime we wanted to apply it, it would not be easy to use. Rather, we will use the definition to compute the Laplace transform for a few basic functions and to establish some general properties that will allow us to build formulas for more complicated functions.

8.2.1. *Linearity.* The most important property of the Laplace transform $\mathcal{L}$ is that it is a linear operator.

Theorem. If $\mathcal{L}[f](s)$ and $\mathcal{L}[g](s)$ exist for some s then so does $\mathcal{L}[f+g](s)$ and $\mathcal{L}[cf](s)$ for every constant c with

$$\mathcal{L}[f+g](s) = \mathcal{L}[f](s) + \mathcal{L}[g](s)\,, \qquad \mathcal{L}[cf](s) = c\mathcal{L}[f](s)\,. \tag{8.3}$$

Proof. This follows directly from definition (8.2) and the facts that definite integrals and limits depend linearly on their arguments. Specifically, we see that

$$\begin{aligned}\mathcal{L}[f+g](s) &= \lim_{T\to\infty}\int_0^T e^{-st}\big(f(t)+g(t)\big)\,\mathrm{d}t \\ &= \lim_{T\to\infty}\int_0^T e^{-st}f(t)\,\mathrm{d}t + \lim_{T\to\infty}\int_0^T e^{-st}g(t)\,\mathrm{d}t = \mathcal{L}[f](s)+\mathcal{L}[g](s)\,, \\ \mathcal{L}[cf](s) &= \lim_{T\to\infty}\int_0^T e^{-st}cf(t)\,\mathrm{d}t = c\lim_{T\to\infty}\int_0^T e^{-st}f(t)\,\mathrm{d}t = c\mathcal{L}[f](s)\,.\end{aligned}$$

□

Example. Compute $\mathcal{L}[\cos(bt)](s)$ and $\mathcal{L}[\sin(bt)](s)$ for any real $b \neq 0$.

Solution. This can be done by using the Euler identity $e^{ibt} = \cos(bt) + i\sin(bt)$ and the linearity (8.3) of $\mathcal{L}$. Then

$$\mathcal{L}[\cos(bt)](s) + i\mathcal{L}[\sin(bt)](s) = \mathcal{L}[e^{ibt}](s) = \frac{1}{s-ib} = \frac{s+ib}{s^2+b^2} \quad \text{for } s > 0\,.$$

By equating the real and imaginary parts above, we see that

$$\begin{aligned}\mathcal{L}[\cos(bt)](s) &= \frac{s}{s^2+b^2} \quad \text{for } s>0\,, \\ \mathcal{L}[\sin(bt)](s) &= \frac{b}{s^2+b^2} \quad \text{for } s>0\,.\end{aligned}$$

8.2.2. *Exponentials and Translations.* Another property of the Laplace transform $\mathcal{L}$ is that it turns multiplication by an exponential in t into a translation of s.

> **Theorem.** If $\mathcal{L}[f](s)$ exists for every $s > \alpha$ and a is any real number then $\mathcal{L}[e^{at}f(t)](s)$ exists for every $s > \alpha + a$ with
>
> $$\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a)\,.$$

Proof. This follows directly from definition (8.2). Specifically, we see that

$$\mathcal{L}[e^{at}f](s) = \lim_{T\to\infty}\int_0^T e^{-st}e^{at}f(t)\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}f(t)\,\mathrm{d}t = \mathcal{L}[f](s-a)\,.$$

□

Examples. From our previous examples and the above theorem we see that

$$\begin{aligned}\mathcal{L}[e^{(a+ib)t}](s) &= \frac{1}{s-a-ib} \quad \text{for } s>a\,, \\ \mathcal{L}[e^{at}\cos(bt)](s) &= \frac{s-a}{(s-a)^2+b^2} \quad \text{for } s>a\,, \\ \mathcal{L}[e^{at}\sin(bt)](s) &= \frac{b}{(s-a)^2+b^2} \quad \text{for } s>a\,.\end{aligned}$$

The Laplace transform also turns a translation of t into multiplication by an exponential in s. Notice that $\mathcal{L}[f](s)$ only depends on the values of $f(t)$ over $[0.\infty)$. Therefore before we translate f we multiply it by the *unit step function* $u(t)$ defined by

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0\,, \\ 0 & \text{for } t < 0\,. \end{cases} \tag{8.4}$$

Because the functions uf and f agree over $[0, \infty)$, it is clear that $\mathcal{L}[uf](s) = \mathcal{L}[f](s)$. We now consider the Laplace transform of $u(t-c)f(t-c)$ for every $c > 0$.

Theorem. If $\mathcal{L}[f](s)$ exists for every $s > \alpha$ and $c > 0$ then $\mathcal{L}[u(t-c)f(t-c)](s)$ exists for every $s > \alpha$ with

$$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}\mathcal{L}[f](s) \quad \text{for } s > \alpha\,.$$

Proof. For every $T > c$ we have

$$\begin{aligned} \int_0^T e^{-st}u(t-c)f(t-c)\,\mathrm{d}t = \int_c^T e^{-st}f(t-c)\,\mathrm{d}t &= e^{-cs}\int_c^T e^{-s(t-c)}f(t-c)\,\mathrm{d}t \\ &= e^{-cs}\int_0^{T-c} e^{-st'}f(t')\,\mathrm{d}t'\,. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}[u(t-c)f(t-c)](s) &= \lim_{T\to\infty}\int_0^T e^{-st}u(t-c)f(t-c)\,\mathrm{d}t \\ &= e^{-cs}\lim_{T\to\infty}\int_0^{T-c} e^{-st}f(t)\,\mathrm{d}t = e^{-cs}\mathcal{L}[f](s) \quad \text{for } s > \alpha\,. \end{aligned}$$

□

8.3. **Existence and Differentiablity of the Transform.** In each of the above examples the definite integrals over $[0, T]$ that appear in the limit (8.2) were proper. Indeed, we were able to evaluate the definite integrals analytically and determine the limit (8.2) for every real s. More generally, from calculus we know that a definite integral over $[0, T]$ is proper whenever its integrand is:

- bounded over $[0, T]$,
- continuous at all but a finite number of points in $[0, T]$.

Such an integrand is said to be *piecewise continuous* over $[0, T]$. Because e^{-st} is a continuous (and therefore bounded) function of t over every $[0, T]$ for each real s, the definite integrals over $[0, T]$ that appear in the limit (8.2) will be proper whenever $f(t)$ is *piecewise continuous* over every $[0, T]$.

Example. The function

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t < \pi\,, \\ \cos(t) & \text{for } t \geq \pi\,, \end{cases}$$

is piecewise continuous over every $[0, T]$ because it is clearly bounded over $[0, \infty)$ and its only discontinuity is at the point $t = \pi$.

Example. The so-called *sawtooth* function

$$f(t) = t - k \quad \text{for } k \le t < k+1 \text{ where } k = 0, 1, 2, 3, \cdots ,$$

is piecewise continuous over every $[0, T]$ because it is clearly bounded over $[0, \infty)$ and has discontinuities at the points $t = 1, 2, 3, \cdots$, only a finite number of which lie in each $[0, T]$.

If we assume that $f(t)$ is piecewise continuous over every $[0, T]$, we still have to give a condition under which the limit (8.2) will exist for certain s. Such a condition is provided by the following definition.

Definition. A function $f(t)$ defined over $[0, \infty)$ is said to be of *exponential order α as $t \to \infty$* provided that for every $\sigma > \alpha$ there exist K_σ and T_σ such that

$$|f(t)| \le K_\sigma e^{\sigma t} \quad \text{for every } t \ge T_\sigma .$$

Roughly speaking, a function is of exponential order α as $t \to \infty$ if its absolute value does not grow faster than $e^{\sigma t}$ as $t \to \infty$ for every $\sigma > \alpha$.

Example. The function e^{at} is of exponential order a as $t \to \infty$ because (8.5) holds with $K_\sigma = 1$ and $T_\sigma = 0$ for every $\sigma > a$.

Example. The function $\cos(bt)$ is of exponential order 0 as $t \to \infty$ because (8.5) holds with $K_\sigma = 1$ and $T_\sigma = 0$ for every $\sigma > 0$.

Example. For every $p > 0$ the function t^p is of exponential order 0 as $t \to \infty$. Indeed, for every $\sigma > 0$ the function $e^{-\sigma t} t^p$ takes on its maximum over $[0, \infty)$ at $t = p/\sigma$, whereby

$$e^{-\sigma t} t^p \le \left(\frac{p}{e\sigma} \right)^p \quad \text{for every } t \ge 0 .$$

Therefore (8.5) holds with $K_\sigma = (\frac{p}{e\sigma})^p$ and $T_\sigma = 0$ for every $\sigma > 0$.

It can be shown that if functions f and g are of exponential orders α and β respectively as $t \to \infty$ then the function $f + g$ is of exponential order $\max\{\alpha, \beta\}$ as $t \to \infty$, while the function fg is of exponential order $\alpha + \beta$ as $t \to \infty$.

Example. For every real a the function $e^{at} + e^{-at}$ is of exponential order $|a|$ as $t \to \infty$. This is because the functions e^{at} and e^{-at} are exponential orders a and $-a$ respectively as $t \to \infty$, and because $|a| = \max\{a, -a\}$.

Example. For every $p > 0$ and every real a and b the function $t^p e^{at} \cos(bt)$ is of exponential order a as $t \to \infty$. This is because the functions t^p, e^{at}, and $\cos(bt)$ are of exponential orders 0, a, and 0 respectively as $t \to \infty$.

The fact you should know about the existence of the Laplace transform for certain s is the following.

Theorem. Let $f(t)$ be

- piecewise continuous over every $[0, T]$,
- of exponential order α as $t \to \infty$.

Then for every positive integer k the function $t^k f(t)$ has these same properties. The function $F(s) = \mathcal{L}[f](s)$ is defined for every $s > \alpha$. Moreover, $F(s)$ is infinitely differentiable over $s > \alpha$ with

$$\mathcal{L}[t^k f(t)](s) = (-1)^k F^{(k)}(s) \quad \text{for } s > \alpha\,. \tag{8.5}$$

Proof. Formula (8.5) can be derived formally by differentiating the integrands:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,,\\
F'(s) &= -\int_0^\infty t\,e^{-st} f(t)\,\mathrm{d}t\,,\\
F''(s) &= \int_0^\infty t^2 e^{-st} f(t)\,\mathrm{d}t\,,\\
&\;\vdots\\
F^{(k)}(s) &= (-1)^k \int_0^\infty t^k e^{-st} f(t)\,\mathrm{d}t\,.
\end{aligned}$$

□

Remark. A correct proof would require a justification of taking the derivatives inside the above imporper integrals. We will not go into those details here. However, we will give an easier proof of the fact that $F(s)$ is defined for $s > \alpha$. The proof uses the direct comparison test for the convergence of improper integrals. That test implies that if $g(t)$ and $G(t)$ are piecewise continuous over every $[0, T]$ such that $|g(t)| \leq G(t)$ for every $t \geq 0$ then

$$\int_0^\infty G(t)\,\mathrm{d}t \quad \text{converges} \qquad \Longrightarrow \qquad \int_0^\infty g(t)\,\mathrm{d}t \quad \text{converges}\,.$$

Let $s > \alpha$ and apply this test to $g(t) = e^{-st} f(t)$. Pick σ so that $\alpha < \sigma < s$. Because $f(t)$ is of exponential order α as $t \to \infty$ and $\sigma > \alpha$ there exist K_σ and T_σ such that (8.5) holds. Because $g(t) = e^{-st} f(t)$ is bounded over $[0, T_\sigma]$ there exists B_σ such that $|g(t)| \leq B_\sigma$ over $[0, T_\sigma]$. It thereby follows that

$$|g(t)| = e^{-st}|f(t)| \leq G(t) \equiv \begin{cases} B_\sigma & \text{for } 0 \leq t < T_\sigma \\ K_\sigma e^{(\sigma - s)t} & \text{for } t \geq T_\sigma\,. \end{cases}$$

Because $s > \sigma$ for this $G(t)$ it can be shown that

$$\int_0^\infty G(t)\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T G(t)\,\mathrm{d}t \quad \text{converges}\,.$$

It follows that the limit in (8.2) converges, whereby $F(s) = \mathcal{L}[f](s)$ is defined at s.

Example. Because for every real a and b we have

$$\mathcal{L}[e^{(a+ib)t}](s) = \frac{1}{s - a - ib} \quad \text{for } s > a\,,$$

it follows from the above theorem that for every nonnegative integer k

$$\mathcal{L}[t^k e^{(a+ib)t}](s) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \frac{1}{s-a-ib} = \frac{k!}{(s-a-ib)^{k+1}} \quad \text{for } s > a\,.$$

This formula implies that for every real a and b and every nonnegative integer k

$$\begin{aligned}
\mathcal{L}[t^k](s) &= \frac{k!}{s^{k+1}} \quad \text{for } s > 0\,,\\
\mathcal{L}[t^k e^{at}](s) &= \frac{k!}{(s-a)^{k+1}} \quad \text{for } s > a\,.\\
\mathcal{L}[t^k e^{at}\cos(bt)](s) &= \mathrm{Re}\left(\frac{k!}{(s-a-ib)^{k+1}}\right) \quad \text{for } s > a\,,\\
\mathcal{L}[t^k e^{at}\sin(bt)](s) &= \mathrm{Im}\left(\frac{k!}{(s-a-ib)^{k+1}}\right) \quad \text{for } s > a\,.
\end{aligned}$$

8.4. **Transform of Dervatives.** The previous result shows that the Laplace transform turns a multiplication by t into a derivative with respect to s. The next result shows the Laplace transform turns a derivative with respect to t into a multiplication by s.

Theorem. Let $f(t)$ be continuous over $[0,\infty)$ such that

- $f(t)$ is of exponential order α as $t \to \infty$,
- $f'(t)$ is piecewise continuous over every $[0,T]$.

Then $\mathcal{L}[f'](s)$ is defined for every $s > \alpha$ with

$$\mathcal{L}[f'](s) = s\,\mathcal{L}[f](s) - f(0)\,.$$

Proof. Let $s > \alpha$. By definition (8.2), an integration by parts, the fact that $f(t)$ is of exponential order α as $t \to \infty$, and the fact that $\mathcal{L}[f](s)$ exists, we see that

$$\begin{aligned}
\mathcal{L}[f'](s) &= \lim_{T\to\infty} \int_0^T e^{-st} f'(t)\,\mathrm{d}t = \lim_{T\to\infty} \left[e^{-st} f(t)\Big|_{t=0}^{T} + s \int_0^T e^{-st} f(t)\,\mathrm{d}t \right]\\
&= \lim_{T\to\infty} e^{-sT} f(T) - f(0) + s \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t = -f(0) + s\,\mathcal{L}[f](s)\,.
\end{aligned}$$

□

If $f(t)$ is sufficiently differentiable then the previous result can be applied repeatedly. For example, if $f(t)$ is twice differentiable then

$$\begin{aligned}
\mathcal{L}[f''](s) = s\,\mathcal{L}[f'](s) - f'(0) &= s\big(s\,\mathcal{L}[f](s) - f(0)\big) - f'(0)\\
&= s^2\mathcal{L}[f](s) - s\,f(0) - f'(0)\,.
\end{aligned}$$

If $f(t)$ is thrice differentiable then

$$\begin{aligned}
\mathcal{L}[f'''](s) = s\,\mathcal{L}[f''](s) - f''(0) &= s\big(s^2\mathcal{L}[f](s) - s\,f(0) - f'(0)\big) - f''(0)\\
&= s^3\mathcal{L}[f](s) - s^2 f(0) - s\,f'(0) - f''(0)\,.
\end{aligned}$$

Proceeding in this way we can use induction to prove the following.

Theorem. Let $f(t)$ be n-times differentiable over $[0,\infty)$ such that

- $f(t)$, $f'(t)$, $\cdots$, $f^{(n-1)}(t)$ are of exponential order α as $t\to\infty$,
- $f^{(n)}(t)$ is piecewise continuous over every interval $[0,T]$.

Then $\mathcal{L}[f^{(n)}](s)$ is defined for every $s>\alpha$ with

$$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0)\,. \tag{8.6}$$

This means that if we know that a function $y(t)$ is n-times differentiable and that it and its first $n-1$ derivatives are of exponential order as $t\to\infty$ then if $Y(s)=\mathcal{L}[y](s)$ we have

$$\begin{aligned}
\mathcal{L}[y'](s) &= s\,Y(s) - y(0)\,,\\
\mathcal{L}[y''](s) &= s^2Y(s) - s\,y(0) - y'(0)\,,\\
\mathcal{L}[y'''](s) &= s^3Y(s) - s^2y(0) - s\,y'(0) - y''(0)\,,\\
&\ \vdots\\
\mathcal{L}[y^{(n)}](s) &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - s\,y^{(n-2)}(0) - y^{(n-1)}(0)\,.
\end{aligned} \tag{8.7}$$

8.5. **Application to Initial-Value Problems.** Because the Laplace transform turns derivatives with respect to t into multiplications by s, it transforms initial-value problems into algebraic problems. Suppose that $y(t)$ is the solution of the initial-value problem

$$\begin{gathered}
y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = f(t)\,,\\
y(0)=y_0\,,\qquad y'(0)=y_1\,,\qquad \cdots\qquad y^{(n-1)}(0)=y_{n-1}\,.
\end{gathered}$$

It can be shown that if $f(t)$ is piecewise continuous over $[0,\infty)$ and is of exponential order as $t\to\infty$ then $y(t)$ is n-times differentiable and that it and its first $n-1$ derivatives are of exponential order as $t\to\infty$. We thereby can use the Laplace transform to find $Y(s)=\mathcal{L}[y](s)$ in terms of the inital data y_0, y_1, $\cdots$, y_{n-1}, and the Laplace transform of the forcing, $F(s)=\mathcal{L}[f](s)$. Later we will see how to determine $y(t)$ from $Y(s)$, but here we will illustrate how to compute $Y(s)$.

First, we use the linearity of $\mathcal{L}$ to express the Laplace transform of the initial-value problem as

$$\mathcal{L}[y^{(n)}] + a_1\mathcal{L}[y^{(n-1)}] + \cdots + a_{n-1}\mathcal{L}[y'] + a_n\mathcal{L}[y] = \mathcal{L}[f]\,.$$

Second, we use (8.7) and the initial conditions to write

$$\begin{aligned}
\mathcal{L}[y](s) &= Y(s)\,,\\
\mathcal{L}[y'](s) &= s\,Y(s) - y_0\,,\\
\mathcal{L}[y''](s) &= s^2Y(s) - s\,y_0 - y_1\,,\\
&\ \vdots\\
\mathcal{L}[y^{(n)}](s) &= s^nY(s) - s^{n-1}y_0 - s^{n-2}y_1 - \cdots - s\,y_{n-1} - y_{n-1}\,.
\end{aligned}$$

Third, we compute $F(s) = \mathcal{L}[f](s)$. Upon combining these, we see that $Y(s)$ satisfies the linear algebraic equation

$$p(s)Y(s) = q(s) + F(s)\,,$$

where $p(s)$ is the characteristic polynomial

$$p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n\,.$$

and $q(s)$ is the polynomial given in terms of the initial data by

$$\begin{aligned} q(s) = {}& \big(s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-2}s + a_{n-1}\big)y_0 \\ &+ \big(s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-3}s + a_{n-2}\big)y_1 \\ &+ \cdots + \big(s^2 + a_1 s + a_2\big)y_{n-3} + \big(s + a_1\big)y_{n-2} + y_{n-1}\,. \end{aligned}$$

Finally, we solve the linear algebraic equation for $Y(s)$ to obtain

$$Y(s) = \frac{q(s) + F(s)}{p(s)}\,. \tag{8.8}$$

The hardest step to find $Y(s)$ given by (8.8) is computing $F(s) = \mathcal{L}[f](s)$. Usually $f(t)$ can be expressed as a combination of the basic forms whose Laplace transform we have already computed. Some of these basic forms are

$$\begin{aligned} \mathcal{L}[t^n](s) &= \frac{n!}{s^{n+1}} && \text{for } s > 0\,, \\ \mathcal{L}[\cos(bt)](s) &= \frac{s}{s^2 + b^2} && \text{for } s > 0\,, \\ \mathcal{L}[\sin(bt)](s) &= \frac{b}{s^2 + b^2} && \text{for } s > 0\,, \\ \mathcal{L}[e^{at}j(t)](s) &= J(s - a) && \text{where } J(s) = \mathcal{L}[j(t)](s)\,, \\ \mathcal{L}[t^n j(t)](s) &= (-1)^n J^{(n)}(s) && \text{where } J(s) = \mathcal{L}[j(t)](s)\,, \\ \mathcal{L}[u(t-c)j(t-c)](s) &= e^{-cs}J(s) && \text{where } J(s) = \mathcal{L}[j(t)](s) \text{ and} \\ & && u(t) \text{ is the unit step function}\,, \\ \mathcal{L}[e^{at}t^n](s) &= \frac{n!}{(s-a)^{n+1}} && \text{for } s > a\,, \\ \mathcal{L}[e^{at}\cos(bt)](s) &= \frac{s - a}{(s-a)^2 + b^2} && \text{for } s > a\,, \\ \mathcal{L}[e^{at}\sin(bt)](s) &= \frac{b}{(s-a)^2 + b^2} && \text{for } s > a\,. \end{aligned} \tag{8.9}$$

We can use these to build up a much longer table of basic forms like the table given in the book. However, the above table contains all the forms we really need. In fact, we can argue the first three entries are redundent because they follow by setting $a = 0$

in the last three. Alternatively, we can argue that the last three entries are redundent because they follow immediately from the first three and the fourth. On exams you will be given a table that includes at least the last six entries above, so there is no need to memorize this table. However, you should learn how to use it efficiently.

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y' - 2y = e^{5t}\,, \qquad y(0) = 3\,.$$

Solution. By setting $a = 5$ and $n = 0$ in the seventh entry of table (8.9) we see that $\mathcal{L}[e^{5t}](s) = 1/(s-5)$. Therefore the Laplace transform of the initial-value problem is

$$\mathcal{L}[y'](s) - 2\mathcal{L}[y](s) = \mathcal{L}[e^{5t}](s) = \frac{1}{s-5}\,,$$

where we see from (8.7) that

$$\mathcal{L}[y](s) = Y(s)\,, \qquad \mathcal{L}[y'](s) = s\,Y(s) - y(0) = s\,Y(s) - 3\,.$$

It follows that

$$(s-2)Y(s) - 3 = \frac{1}{s-5}\,, \qquad \Longrightarrow \qquad Y(s) = \frac{1}{(s-2)(s-5)} + \frac{3}{s-2}\,.$$

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' - 2y' - 8y = 0\,, \qquad y(0) = 3\,, \quad y'(0) = 7\,.$$

Solution. Here there is no forcing. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) - 2\mathcal{L}[y'](s) - 8\mathcal{L}[y](s) = 0\,,$$

where we see from (8.7) that

$$\begin{aligned}
\mathcal{L}[y](s) &= Y(s)\,,\\
\mathcal{L}[y'](s) &= s\,Y(s) - y(0) = s\,Y(s) - 3\,,\\
\mathcal{L}[y''](s) &= s^2Y(s) - s\,y(0) - y'(0) = s^2Y(s) - 3s - 7\,.
\end{aligned}$$

It follows that

$$(s^2 - 2s - 8)Y(s) - 3s - 1 = 0\,, \qquad \Longrightarrow \qquad Y(s) = \frac{3s+1}{s^2 - 2s - 8}\,.$$

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y = \sin(3t)\,, \qquad y(0) = y'(0) = 0\,.$$

Solution. By setting $b = 3$ in the third entry of table (8.9) we see that $\mathcal{L}[\sin(3t)](s) = 3/(s^2+9)$. Therefore the Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = \mathcal{L}[\sin(3t)](s) = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}\,,$$

where we see from (8.7) that

$$\mathcal{L}[y](s) = Y(s)\,, \qquad \mathcal{L}[y''](s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)\,.$$

It follows that

$$(s^2+4)Y(s) = \frac{3}{s^2+9}\,, \qquad \Longrightarrow \qquad Y(s) = \frac{3}{(s^2+4)(s^2+9)}\,.$$

8.6. **Piecewise-Defined Forcing.** The Laplace transform method can be to solve initial-value problems of the form

$$\begin{gathered} y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f(t)\,, \\ y(0) = y_0\,, \qquad y'(0) = y_1\,, \qquad \cdots \qquad y^{(n-1)}(0) = y_{n-1}\,, \end{gathered}$$

where the forcing $f(t)$ is piecewise-defined over $[0,\infty)$ by a list of cases given in the form

$$f(t) = \begin{cases} f_0(t) & \text{for } 0 \le t < c_1\,, \\ f_1(t) & \text{for } c_1 \le t < c_2\,, \\ \vdots & \qquad\vdots \\ f_{m-1}(t) & \text{for } c_{m-1} \le t < c_m\,, \\ f_m(t) & \text{for } c_m \le t < \infty\,, \end{cases}$$

where $0 = c_0 < c_1 < c_2 < \cdots < c_m < \infty$. We assume that for each $k = 0,\, 1,\, \cdots,\, m-1$ the function f_k is continuous and bounded over $[c_k, c_{k+1})$, while the function f_m is continuous over $[c_m,\infty)$ and is of exponential order as $t \to \infty$. Here we show how to compute the Laplace transform $F(s) = \mathcal{L}[f](s)$ for such a function. There are three steps.

The first step is to express $f(t)$ in terms of translations of the unit step $u(t)$. How this is done should become clear once you see that for every $0 \le c < d$ we have

$$u(t-c) - u(t-d) = \begin{cases} 1 & \text{for } c \le t < d\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

In other words, the function $u(t-c) - u(t-d)$ is a switch that turns on at $t = c$ and turns off at $t = d$. So for any given function $g(t)$ we have

$$\big(u(t-c) - u(t-d)\big)\, g(t) = \begin{cases} g(t) & \text{for } c \le t < d\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This observation allows us to express $f(t)$ as

$$\begin{aligned} f(t) = \big(u(t) - u(t-c_1)\big)\, f_0(t) + \big(u(t-c_1) - u(t-c_2)\big)\, f_1(t) \\ + \cdots + \big(u(t-c_{m-1}) - u(t-c_m)\big)\, f_{m-1}(t) + u(t-c_m)\, f_m(t)\,. \end{aligned}$$

By grouping terms above that involve the same $u(t-c_k)$, we bring $f(t)$ into the form

$$f(t) = f_0(t) + u(t-c_1)\, h_1(t) + \cdots + u(t-c_m)\, h_m(t)\,, \tag{8.10}$$

where $h_k(t) = f_k(t) - f_{k-1}(t)$ for $k = 1, 2, \cdots, m$. This is the form we want. You can get to it either by carrying out the grouping indicated above or by memorizing (8.10). It helps to recall that each term $u(t-c_k)\, h_k(t)$ appearing in (8.10) simply changes the forcing from $f_{k-1}(t)$ to $f_k(t)$ at time $t = c_k$ because $h_k(t) = f_k(t) - f_{k-1}(t)$.

The idea of the second step is to bring (8.10) into a form that allows us to use the sixth entry in table (8.9). That entry states that $\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$. Therefore we must recast (8.10) into the form

$$f(t) = f_0(t) + u(t-c_1)\, j_1(t-c_1) + \cdots + u(t-c_m)\, j_m(t-c_m)\,.$$

Each function $j_k(t)$ is obtained from the $h_k(t)$ appearing in (8.10) by

$$j_k(t) = h_k(t+c_k) \quad \text{for } k = 1, 2, \cdots, m\,.$$

Indeed, this formula implies $j_k(t-c_k) = h_k(t)$, which is what appears in (8.10).

Once we have found all the $j_k(t)$ then the final step is to compute $\mathcal{L}[f_0](s)$ and each $\mathcal{L}[j_k](s)$, and use the fact that

$$\mathcal{L}[u(t-c_k)j_k(t-c_k)](s) = e^{-c_k s}\mathcal{L}[j_k](s) \quad \text{for } k = 1, 2, \cdots, m\,,$$

to compute $\mathcal{L}[f](s)$ as

$$\mathcal{L}[f](s) = \mathcal{L}[f_0](s) + e^{-c_1 s}\mathcal{L}[j_1](s) + \cdots + e^{-c_m s}\mathcal{L}[j_m](s)\,.$$

Often we will have to use identities to express $f_0(t)$ and each $j_k(t)$ in forms that allows us to compute their Laplace transforms from table (8.9).

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y = f(t)\,, \qquad y(0) = 7\,, \quad y'(0) = 5\,,$$

where

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 2\,, \\ 2t & \text{for } 2 \le t < 4\,, \\ 4 & \text{for } 4 \le t\,. \end{cases}$$

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = F(s)\,,$$

where $F(s) = \mathcal{L}[f](s)$ and

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 7\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 7s - 5\,. \end{aligned}$$

The Laplace transform of the initial-value problem thereby becomes

$$(s^2+4)Y(s) - 7s - 5 = F(s)\,, \qquad \Longrightarrow \qquad Y(s) = \frac{1}{s^2+4}\big(7s + 5 + F(s)\big)\,.$$

All that remains to be done is to compute $F(s)$. The first step is to express $f(t)$ in terms of unit step functions as

$$\begin{aligned} f(t) &= \big(u(t) - u(t-2)\big)\, t^2 + \big(u(t-2) - u(t-4)\big)\, 2t + u(t-4)\, 4 \\ &= t^2 + u(t-2)\,(2t - t^2) + u(t-4)\,(4 - 2t)\,. \end{aligned}$$

The second step is to write

$$f(t) = t^2 + u(t-2)\, j_1(t-2) + u(t-4)\, j_2(t-4)\,,$$

where

$$j_1(t) = 2(t+2) - (t+2)^2 = 2t + 4 - t^2 - 4t - 4 = -t^2 - 2t\,,$$
$$j_2(t) = 4 - 2(t+4) = -2t - 4\,.$$

Here we obtained $j_1(t)$ by replacing t with $t+2$ in the factor $(2t - t^2)$ and $j_2(t)$ by replacing t with $t+4$ in the factor $(4-2t)$. Finally, the above form for $f(t)$ allows us to use the sixth entry of table (8.9) to compute $F(s) = \mathcal{L}[f](s)$ as

$$\begin{aligned}
F(s) &= \mathcal{L}[t^2](s) + \mathcal{L}[u(t-2)\,j_1(t-2)](s) + \mathcal{L}[u(t-4)\,j_2(t-4)](s) \\
&= \mathcal{L}[t^2](s) - e^{-2s}\mathcal{L}[t^2 + 2t](s) - e^{-4s}\mathcal{L}[2t+4](s) \\
&= \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) - e^{-4s}\left(\frac{2}{s^2} + \frac{4}{s}\right) \\
&= \left(1 - e^{-2s}\right)\frac{2}{s^3} - \left(e^{-2s} + e^{-4s}\right)\frac{2}{s^2} - e^{-4s}\,\frac{4}{s}\,.
\end{aligned}$$

It follows that

$$Y(s) = \frac{7s+5}{s^2+4} + \left(1 - e^{-2s}\right)\frac{2}{s^3(s^2+4)} - \left(e^{-2s} + e^{-4s}\right)\frac{2}{s^2(s^2+4)} - e^{-4s}\,\frac{4}{s\,(s^2+4)}\,.$$

8.7. **Inverse Transform.** The process of determining $y(t)$ from $Y(s)$ is called taking the inverse Laplace transform. It is important to know that this process has a unique result. Indeed, we will use the following theorem.

Theorem. Let $f(t)$ and $g(t)$ be two functions over $[0,\infty)$ and α a real number such that

- $f(t)$ and $g(t)$ are of exponential order α as $t \to \infty$,
- $f(t)$ and $g(t)$ are piecewise continuous over every $[0,T]$,
- $\mathcal{L}[f](s) = \mathcal{L}[g](s)$ for every $s > \alpha$.

Then $f(t) = g(t)$ for every t in $[0,\infty)$.

The proof of this result requires tools from complex variables that are beyond the scope of this course. Fortunately, you do not need to know how to prove this result to use it! Its usefulness stems from the fact that solutions $y(t)$ to the initial-value problems we are considering lie within the class of functions considered above — namely, they are functions that are of exponential order as $t \to \infty$ and that are piecewise continuous over every $[0,T]$. In fact, they are continuous and piecewise differentiable over every $[0,T]$. This means that if we succeed in finding a function $y(t)$ within this class such that $\mathcal{L}[y](s) = Y(s)$ then it will be the unique solution of the initial-value problem that we seek.

Because the above result states there is a unique $f(t)$ that is of exponential order as $t \to \infty$ and is piecewise continuous over every $[0,T]$ such that $\mathcal{L}[f](s) = F(s)$, we introduce the notation

$$f(t) = \mathcal{L}^{-1}[F](t)\,.$$

The operator $\mathcal{L}^{-1}$ denotes the *inverse Laplace transform.* Because it undoes the Laplace transform $\mathcal{L}$, it inherits many properties from $\mathcal{L}$. For example, it is linear. We can also easily read-off from the first and last three entries in table (8.9) of basic forms that

$$
\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right](t) = t^n\,, && \mathcal{L}^{-1}\left[\frac{n!}{(s-a)^{n+1}}\right](t) = e^{at}t^n\,,\\
&\mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right](t) = \cos(bt)\,, && \mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right](t) = e^{at}\cos(bt)\,,\\
&\mathcal{L}^{-1}\left[\frac{b}{s^2+b^2}\right](t) = \sin(bt)\,, && \mathcal{L}^{-1}\left[\frac{b}{(s-a)^2+b^2}\right](t) = e^{at}\sin(bt)\,.
\end{aligned}
\tag{8.11}
$$

It is also clear from the sixth entry of table (8.9) that

$$
\mathcal{L}^{-1}[e^{-cs}J(s)](t) = u(t-c)\,j(t-c)\,, \quad \text{where } j(t) = \mathcal{L}^{-1}[J](t)\,. \tag{8.12}
$$

For us, the process of computing $y(t) = \mathcal{L}^{-1}[Y](t)$ for a given $Y(s)$ will be one of expressing $Y(s)$ as a sum of terms that will allow us to read off $y(t)$ from the basic forms above. To illustrate this process, we will compute $y(t) = \mathcal{L}^{-1}[Y](t)$ for the $Y(s)$ found in the examples given in the previous section, thereby completing our solution of the initial-value problems.

Example. Find $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$
Y(s) = \frac{1}{(s-2)(s-5)} + \frac{3}{s-2}\,.
$$

Solution. By the partial fraction identity

$$
\frac{1}{(s-2)(s-5)} = \frac{\frac{1}{3}}{s-5} + \frac{-\frac{1}{3}}{s-2}\,,
$$

we can express $Y(s)$ as

$$
Y(s) = \tfrac{1}{3}\frac{1}{s-5} + \tfrac{8}{3}\frac{1}{s-2}\,.
$$

The top right entry of table (8.11) with $a = 5$ and $a = 2$ then yields

$$
y(t) = \mathcal{L}^{-1}[Y(s)](t) = \tfrac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-5}\right](t) + \tfrac{8}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right](t) = \tfrac{1}{3}e^{5t} + \tfrac{8}{3}e^{2t}\,.
$$

Example. Find $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$
Y(s) = \frac{3s+1}{s^2-2s-8}\,.
$$

Solution. By the partial fraction identity

$$
\frac{3s+1}{s^2-2s-8} = \frac{3s+1}{(s-4)(s+2)} = \frac{\frac{13}{6}}{s-4} + \frac{\frac{5}{6}}{s+2}\,,
$$

we can express $Y(s)$ as

$$
Y(s) = \tfrac{13}{6}\frac{1}{s-4} + \tfrac{5}{6}\frac{1}{s+2}\,.
$$

The top right entry of table (8.11) with $a = 4$ and $a = -2$ then yields

$$
y(t) = \mathcal{L}^{-1}[Y(s)](t) = \tfrac{13}{6}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right](t) + \tfrac{5}{6}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t) = \tfrac{13}{6}e^{4t} + \tfrac{5}{6}e^{-2t}\,.
$$

Example. Find $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{3}{(s^2+4)(s^2+9)} \,.$$

Solution. By the partial fraction identity

$$\frac{3}{(z+4)(z+9)} = \frac{\frac{3}{5}}{(z+4)} + \frac{-\frac{3}{5}}{(z+9)} \,,$$

we can express $Y(s)$ as

$$Y(s) = \frac{\frac{3}{5}}{s^2+4} - \frac{\frac{3}{5}}{s^2+9} = \tfrac{3}{10}\frac{2}{s^2+2^2} - \tfrac{1}{5}\frac{3}{s^2+3^2} \,.$$

The bottom left entry of table (8.11) with $b = 2$ and $b = 3$ then yields

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = \tfrac{3}{10}\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right](t) - \tfrac{1}{5}\mathcal{L}^{-1}\left[\frac{3}{s^2+3^2}\right](t) = \tfrac{3}{10}\sin(2t) - \tfrac{1}{5}\sin(3t) \,.$$

Example. Find $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{7s+5}{s^2+4} + \left(1 - e^{-2s}\right)\frac{2}{s^3(s^2+4)} - \left(e^{-2s} + e^{-4s}\right)\frac{2}{s^2(s^2+4)} - e^{-4s}\frac{4}{s\,(s^2+4)} \,.$$

Solution. We first derive the partial fraction identities

$$\frac{7s+5}{s^2+4} = \frac{7s}{s^2+4} + \frac{5}{s^2+4} \,, \qquad \frac{2}{s^2(s^2+4)} = \frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+4} \,,$$

$$\frac{2}{s^3(s^2+4)} = \frac{\frac{1}{2}}{s^3} - \frac{\frac{1}{8}}{s} + \frac{\frac{1}{8}s}{s^2+4} \,, \qquad \frac{4}{s\,(s^2+4)} = \frac{1}{s} - \frac{s}{s^2+4} \,.$$

The top left of these is straightforward. The top right identity only involves s^2, so is simply the identity

$$\frac{2}{z(z+4)} = \frac{\frac{1}{2}}{z} - \frac{\frac{1}{2}}{z+4} \,, \quad \text{evaluated at } z = s^2 \,.$$

The bottom right identity is simply $2s$ times the top right one. Finally, the bottom left identity is obtained by first dividing the top right one by s and then employing the bottom right one divided by 8 to the last term.

These partial fraction identites allow us to express $Y(s)$ as

$$\begin{aligned} Y(s) &= \frac{7s}{s^2+4} + \frac{5}{s^2+4} + \left(1 - e^{-2s}\right)\left(\frac{\frac{1}{2}}{s^3} - \frac{\frac{1}{8}}{s} + \frac{\frac{1}{8}s}{s^2+4}\right) \\ &\quad - \left(e^{-2s} + e^{-4s}\right)\left(\frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+4}\right) - e^{-4s}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) \\ &= 7\frac{s}{s^2+2^2} + \frac{5}{2}\frac{2}{s^2+2^2} + \left(1 - e^{-2s}\right)\left(\frac{1}{4}\frac{2}{s^3} - \frac{1}{8}\frac{1}{s} + \frac{1}{8}\frac{s}{s^2+2^2}\right) \\ &\quad - \left(e^{-2s} + e^{-4s}\right)\left(\frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{2}{s^2+2^2}\right) - e^{-4s}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right) . \end{aligned}$$

The formulas in the first column of table (8.11) show that

$$\begin{aligned}
\mathcal{L}^{-1}\bigg[7\frac{s}{s^2+2^2}+\frac{5}{2}\frac{2}{s^2+2^2}\bigg](t) &= 7\cos(2t)+\tfrac{5}{2}\cos(2t)\,,\\
\mathcal{L}^{-1}\bigg[\frac{1}{4}\frac{2}{s^3}-\frac{1}{8}\frac{1}{s}+\frac{1}{8}\frac{s}{s^2+2^2}\bigg](t) &= \tfrac{1}{4}t^2-\tfrac{1}{8}+\tfrac{1}{8}\cos(2t)\,,\\
\mathcal{L}^{-1}\bigg[\frac{1}{2}\frac{1}{s^2}-\frac{1}{4}\frac{2}{s^2+2^2}\bigg](t) &= \tfrac{1}{2}t-\tfrac{1}{4}\sin(2t)\,,\\
\mathcal{L}^{-1}\bigg[\frac{1}{s}-\frac{s}{s^2+2^2}\bigg](t) &= 1-\cos(2t)\,.
\end{aligned}$$

By combining these facts with formula (8.12), it follows that

$$\begin{aligned}
y(t) = 7\cos(2t) &+ \tfrac{5}{2}\cos(2t)+\big(\tfrac{1}{4}t^2-\tfrac{1}{8}+\tfrac{1}{8}\cos(2t)\big)\\
&- u(t-2)\big(\tfrac{1}{4}(t-2)^2-\tfrac{1}{8}+\tfrac{1}{8}\cos(2(t-2))\big)\\
&- u(t-2)\big(\tfrac{1}{2}(t-2)-\tfrac{1}{4}\sin(2(t-2))\big)\\
&- u(t-4)\big(\tfrac{1}{2}(t-4)-\tfrac{1}{4}\sin(2(t-4))\big)\\
&- u(t-4)\big(1-\cos(2(t-4))\big)\,.
\end{aligned}$$

8.8. **Computing Green Functions.** The Laplace transform can be used to efficiently compute Green functions for differential operators with constant coefficients. Recall that given the n^{th}-order differential operator L with constant coefficients given by

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,,$$

the Green function $g(t)$ associated with L is the solution of the initial-value problem

$$\begin{gathered}
g^{(n)} + a_1 g^{(n-1)} + \cdots + a_{n-1}g' + a_n g = 0\,,\\
g(0)=0\,,\quad g'(0)=0\,,\quad \cdots\quad g^{(n-2)}(0)=0\,,\quad g^{(n-1)}(0)=1\,.
\end{gathered}$$

The Laplace transform of this initial-value problem is

$$\mathcal{L}\big[g^{(n)}\big](s) + a_1\mathcal{L}\big[g^{(n-1)}\big](s) + \cdots + a_{n-1}\mathcal{L}[g'](s) + \mathcal{L}[g](s) = 0\,,$$

where if $G(s) = \mathcal{L}[g](s)$ then

$$\begin{aligned}
\mathcal{L}[g'](s) &= s\,G(s) - g(0) = s\,G(s)\,,\\
\mathcal{L}\big[g''\big](s) &= s^2G(s) - s\,g(0) - g'(0) = s^2G(s)\,,\\
&\vdots\\
\mathcal{L}\big[g^{(n-1)}\big](s) &= s^{n-1}G(s) - s^{n-2}g(0) - s^{n-3}g'(0) - \cdots - g^{(n-2)}(0) = s^{n-1}G(s)\,,\\
\mathcal{L}\big[g^{(n)}\big](s) &= s^nG(s) - s^{n-1}g(0) - s^{n-2}g'(0) - \cdots - g^{(n-1)}(0) = s^nG(s) - 1\,.
\end{aligned}$$

We thereby see that $G(s)$ satisfies

$$p(s)G(s) - 1 = 0\,, \qquad \Longrightarrow \qquad G(s) = \frac{1}{p(s)}\,,$$

where $p(s)$ is the characteristic polynomial of L, which is given by

$$p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \,.$$

In other words, the Laplace transform of the Green function of L is the reciprocal of the characteristic polynomial of L.

The problem of computing a Green function is thereby reduced to the problem of finding an inverse Laplace transform. This can often be done quickly.

Example. Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^2 + 6\mathrm{D} + 13$.

Solution. Because $p(s) = s^2 + 6s + 13 = (s+3)^2 + 2^2$, the bottom right entry of table (8.11) with $a = -3$ and $b = 2$ shows that the Green function is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+3)^2 + 2^2}\right] = \frac{1}{2}e^{-3t}\sin(2t) \,.$$

Example. Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^2 + 2\mathrm{D} - 15$.

Solution. Because $p(s) = s^2 + 2s - 15 = (s-3)(s+5)$, we use the partial fraction identity

$$\frac{1}{(s-3)(s+5)} = \frac{\frac{1}{8}}{s-3} - \frac{\frac{1}{8}}{s+5} \,.$$

The top right entry of table (8.11) with $a = 3$ and with $a = -5$ shows that the Green function is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right] = \frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s+5}\right] = \frac{e^{3t} - e^{-5t}}{8} \,.$$

Example. Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^4 + 13\mathrm{D}^2 + 36$.

Solution. Because $p(s) = s^4 + 13s^2 + 36 = (s^2+4)(s^2+9)$ only depends on s^2, we can use the partial fraction identity

$$\frac{1}{(z+4)(z+9)} = \frac{\frac{1}{5}}{z+4} - \frac{\frac{1}{5}}{z+9} \quad \text{at } z = s^2 \,.$$

The bottom left entry of table (8.11) with $b = 2$ and with $b = 3$ shows that the Green function is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right] = \frac{1}{10}\mathcal{L}^{-1}\left[\frac{2}{s^2 + 2^2}\right] - \frac{1}{15}\mathcal{L}^{-1}\left[\frac{3}{s^2 + 3^2}\right] = \frac{\sin(2t)}{10} - \frac{\sin(3t)}{15} \,.$$

8.9. **Convolutions.** Let $f(t)$ and $g(t)$ be any two functions defined over the interval $[0, \infty)$. Their *convolution* is a third function $(f * g)(t)$ that is defined by the formula

$$(f * g)(t) = \int_0^t f(t - \tau) g(\tau) \, \mathrm{d}\tau \,, \tag{8.13}$$

whenever the above integral makes sense for every $t \geq 0$. In particular, the convolution of f and g will be defined whenever both f and g are piecewise continuous over every $[0, T]$.

The convolution can be thought of a some kind of product between two functions. It is easily checked that this so-called convolution product satisfies some of the properties of ordinary multipulcation. For example, for any functions f, g, amd h that are piecewise continuous over every $[0,T]$ we have

$$
\begin{aligned}
g * f &= f * g && \text{commutative law}\,,\\
h * (f+g) &= h * f + h * g && \text{distributive law}\,,\\
h * (g * f) &= (h * g) * f && \text{associative law}\,.
\end{aligned}
$$

The commutative law is proved by introducing $\tau' = t-\tau$ as a new variable of integration, whereby one sees that

$$(g * f)(t) = \int_0^t g(t-\tau)f(\tau)\,\mathrm{d}\tau = \int_0^t g(\tau')f(t-\tau')\,\mathrm{d}\tau' = (f * g)(t)\,.$$

Verification of the distributive and associative laws is left to you.

The convolution differs from ordinary multipulcation in some respects too. For example, it is not generaly true that $f * 1 = f$ or that $f * f \geq 0$. Indeed, one sees that

$$(1 * 1)(t) = \int_0^t 1 \cdot 1\,\mathrm{d}\tau = t \neq 1\,,$$

and that

$$
\begin{aligned}
(\sin * \sin)(t) &= \int_0^t \sin(t-\tau)\sin(\tau)\,\mathrm{d}\tau\\
&= \sin(t)\int_0^t \cos(\tau)\sin(\tau)\,\mathrm{d}\tau + \cos(t)\int_0^t \sin(\tau)^2\,\mathrm{d}\tau\\
&= \tfrac{1}{2}\sin(t)^3 + \tfrac{1}{2}t\cos(t) - \tfrac{1}{2}\sin(t)\cos(t)^2 \ngeq 0 \quad \text{for every } t > 0\,.
\end{aligned}
$$

In fact, one can show that $1 * f = f$ if and only if $f = 0$.

The main result of this section is that the Laplace transform of a convolution of two functions is the ordinary product of their Laplace transforms. In other words, the Laplace transform maps convolutions to multiplication.

Convolution Theorem. Let $f(t)$ and $g(t)$ be

- piecewise continuous over every $[0,T]$
- of exponential order α as $t \to \infty$.

Then $\mathcal{L}[f * g](s)$ is defined for every $s > \alpha$ with

$$\mathcal{L}[f * g](s) = F(s)G(s)\,, \quad \text{where } F(s) = \mathcal{L}[f](s) \text{ and } G(s) = \mathcal{L}[g](s)\,. \tag{8.14}$$

Proof. For every $T > 0$ definition (8.13) of convolution implies that

$$\int_0^T e^{-st}\,(f * g)(t)\,\mathrm{d}t = \int_0^T e^{-st}\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau\,\mathrm{d}t = \int_0^T\int_0^t e^{-st}f(t-\tau)g(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\,.$$

We now exchange the order of the definite integrals over τ and t on the right-hand side. As you recall from multivariable Calculus, this should be done carefully because the upper endpoint of the inner integral depends on the variable of integration t of the outer integral. When viewed in the (τ,t)-plane, the domain over which the double integral is being taken is the triangle given by $0 \le \tau \le t \le T$. In general, when the order of definite integrals is exchanged over this domain we have

$$\int_0^T \int_0^t \bullet \; \mathrm{d}\tau \, \mathrm{d}t = \int_0^T \int_\tau^T \bullet \; \mathrm{d}t \, \mathrm{d}\tau \,,$$

where $\bullet$ denotes any appropriate integrand. We thereby obtain

$$\int_0^T e^{-st} \, (f * g)(t) \, \mathrm{d}t = \int_0^T \int_\tau^T e^{-st} f(t-\tau) g(\tau) \, \mathrm{d}t \, \mathrm{d}\tau \,.$$

We now factor e^{-st} as $e^{-st} = e^{-s(t-\tau)} e^{-s\tau}$, and group the factor $e^{-s(t-\tau)}$ with $f(t-\tau)$ and the factor $e^{-s\tau}$ with $g(\tau)$, whereby

$$\begin{aligned} \int_0^T e^{-st} \, (f * g)(t) \, \mathrm{d}t &= \int_0^T \int_\tau^T e^{-s(t-\tau)} f(t-\tau) \, e^{-s\tau} g(\tau) \, \mathrm{d}t \, \mathrm{d}\tau \\ &= \int_0^T e^{-s\tau} g(\tau) \int_\tau^T e^{-s(t-\tau)} f(t-\tau) \, \mathrm{d}t \, \mathrm{d}\tau \,. \end{aligned}$$

We then make the change of variable $t' = t - \tau$ in the inner definite integral to obtain

$$\int_0^T e^{-st} \, (f * g)(t) \, \mathrm{d}t = \int_0^T e^{-s\tau} g(\tau) \int_0^{T-\tau} e^{-st'} f(t') \, \mathrm{d}t' \, \mathrm{d}\tau \,.$$

Upon formally letting $T \to \infty$ above, definition (8.2) of the Laplace transform shows that the inner integral converges to $F(s)$, which is independent of τ. The double integral thereby converges to $G(s)F(s)$, yielding (8.14). $\square$

Remark. Because the upper endpoint of the inner integral depends on the variable of integration τ of the outer integral, properly passing to the limit above requires greater care than we took here. The techniques one needs are taught in Advanced Calculus courses. The argument given above suits our purposes because it illuminates why (8.14) holds.

The convolution theorem can be used to help evaluate inverse Laplace transforms. For example, suppose that we know for a given $F(s)$ and $G(s)$ that $f(t) = \mathcal{L}^{-1}[F](t)$ and $g(t) = \mathcal{L}^{-1}[G](t)$. Then (8.14) implies that

$$\mathcal{L}^{-1}\big[F(s)G(s)\big](t) = (f * g)(t) \,. \tag{8.15}$$

You can use this fact to express inverse Laplace transforms as convolutions. You may still have to evaluate the convolution integral, but some of you might find that easier than using partial fraction identities to express $F(s)G(s)$ in basic forms.

Example. Compute $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{2}{s^2(s^2+4)} \,.$$

Solution. Because we know from table (8.11) that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t \,, \qquad \mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right] = \sin(2t) \,,$$

it follows from (8.15) and an integration by parts that

$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{s^2(s^2+4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{2}{s^2+2^2}\right] = \int_0^t (t-\tau)\sin(2\tau)\,\mathrm{d}\tau$$
$$= (\tau - t)\frac{\cos(2\tau)}{2}\bigg|_0^t - \int_0^t \frac{\cos(2\tau)}{2}\,\mathrm{d}\tau = \frac{t}{2} - \frac{\sin(2t)}{4}\,.$$

This is the same result we got on page 18 using a partial fraction identity.

8.10. **Green Functions and Natural Fundamental Sets.** The convolution theorem gives us other ways to understand Green functions. We have used the Green function to construct a particular solution of the nonhomogeneous equation $\mathrm{L}y = p(\mathrm{D}) = f(t)$ by the formula

$$y_P(t) = \int_0^t g(t-\tau)f(\tau)\,\mathrm{d}\tau\,.$$

Notice that the right-hand side above is exactly $(g*f)(t)$. Taking the Laplace transform of this formula, the Convolution Theorem then yields

$$\mathcal{L}[y_P](s) = \mathcal{L}[g*f](s) = G(s)F(s) = \frac{F(s)}{p(s)}\,, \qquad \text{where } F(s) = \mathcal{L}[f](s)\,.$$

But this agrees with formula (8.8). Indeed, because $y_P(t)$ given by the above formula satisfies the initial conditions

$$y_P(0) = 0\,, \quad y_P'(0) = 0\,, \quad \cdots \quad y_P^{(n-2)}(0) = 0\,, \quad y_P^{(n-1)}(0) = 0\,,$$

it follows that the polynomial $q(s)$ appearing in (8.8) vanishes.

Formula (8.8) can generally be recast as

$$y(t) = y_H(t) + y_P(t)\,, \quad \text{where} \quad y_H(t) = \mathcal{L}^{-1}\left[\frac{q(s)}{p(s)}\right](t)\,, \quad y_P(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{p(s)}\right](t)\,.$$

This is the decomposition of $y(t)$ into the solution $y_H(t)$ of the associated homogeneous equation whose initial data agree with $y(t)$ and the particular solution $y_P(t)$ whose initial data are zero. Because $G(s) = 1/p(s)$,

$$\begin{aligned} q(s) = {} & \big(s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-3}s^2 + a_{n-2}s + a_{n-1}\big)y_0 \\ & + \big(s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-3}s + a_{n-2}\big)y_1 \\ & \vdots \\ & + \big(s^2 + a_1 s + a_2\big)y_{n-3} + \big(s + a_1\big)y_{n-2} + y_{n-1}\,, \end{aligned}$$

and because $\mathrm{D}^k g(t) = \mathcal{L}^{-1}[s^k G(s)](t) = \mathcal{L}^{-1}[s^k/p(s)](t)$ for every $k = 0, 2, \cdots, n-1$, the function $y_H(t) = \mathcal{L}^{-1}[q(s)/p(s)](t)$ can be expressed in terms of $g(t)$ as

$$\begin{aligned} y_H(t) = y_0 &\Big(\mathrm{D}^{n-1} + a_1\mathrm{D}^{n-2} + \cdots + a_{n-3}\mathrm{D}^2 + a_{n-2}\mathrm{D} + a_{n-1}\Big) g(t) \\ &+ y_1\Big(\mathrm{D}^{n-2} + a_1\mathrm{D}^{n-3} + \cdots + a_{n-3}\mathrm{D} + a_{n-2}\Big) g(t) \\ &\vdots \\ &+ y_{n-3}\Big(\mathrm{D}^2 + a_1\mathrm{D} + a_2\Big) g(t) + y_{n-2}\big(\mathrm{D} + a_1\big) g(t) + y_{n-1} g(t) \,. \end{aligned}$$

Therefore the *natural fundamental set of solutions* associated with the homogeneous equation $\mathrm{L}y = 0$ is given in terms of the Green function g by

$$\begin{aligned} N_0(t) &= \Big(\mathrm{D}^{n-1} + a_1\mathrm{D}^{n-2} + \cdots + a_{n-3}\mathrm{D}^2 + a_{n-2}\mathrm{D} + a_{n-1}\Big) g(t) \,, \\ N_1(t) &= \Big(\mathrm{D}^{n-2} + a_1\mathrm{D}^{n-3} + \cdots + a_{n-3}\mathrm{D} + a_{n-2}\Big) g(t) \,, \\ &\vdots \\ N_{n-3}(t) &= \Big(\mathrm{D}^2 + a_1\mathrm{D} + a_2\Big) g(t) \,, \\ N_{n-2}(t) &= \big(\mathrm{D} + a_1\big) g(t) \,, \\ N_{n-1}(t) &= g(t) \,. \end{aligned} \tag{8.16}$$

The solution of the general initial-value problem

$$\mathrm{L}y = f(t) \,, \qquad y(0) = y_0 \,, \quad y'(0) = y_1 \,, \quad \cdots \quad y^{(n-2)}(0) = y_{n-2} \,, \quad y^{(n-1)}(0) = y_{n-1} \,,$$

then can be expressed as

$$y(t) = y_0 N_0(t) + y_1 N_1(t) + \cdots + y_{n-2} N_{n-2}(t) + y_{n-1} N_{n-1}(t) + (N_{n-1} * f)(t) \,,$$

where $N_0(t)$, $N_1(t)$, $\cdots$, $N_{n-1}(t)$ is the natural fundamental set of solutions to the associated homogeneous equation, which is given in terms of the Green function $g(t)$ by (8.16). Recall that $W[N_0, N_1, \cdots, N_{n-1}](0) = 1$, which implies $W[N_0, N_1, \cdots, N_{n-1}](t) = e^{-a_1 t}$ by the Abel formula.