HIGHER-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS III: Mechanical Vibrations

David Levermore
Department of Mathematics
University of Maryland

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Because the presentation of this material in lecture will differ from that in the book, I felt that notes that closely follow the lecture presentation might be appreciated.

7. Mechanical Vibrations

7.1. Spring-Mass Systems. Consider a spring hanging from a support. When an object of mass m is attached to the free end of the spring, the object will eventually come to rest at a lower position. Let y_o and y_r be the vertical rest positions of the free end of the spring without and with the mass attached. We will assume that the mass is constrained to only move vertically and want to describe the vertical postition $y(t)$ of the mass as a function of time t when the mass is initially displaced from y_r, or is given some initial velocity, or is driven by an external force $F_{ext}(t)$.

The forces acting on the mass that we will consider are the gravitational force F_{grav}, the spring force F_{spr}, the damping or drag force F_{damp}, and the external or driving force F_{ext}. Newton's law of motion then states that

$$m\frac{d^2y}{dt^2} = F_{grav} + F_{spr} + F_{damp} + F_{ext}\,. \tag{7.1}$$

Always be sure you are working in one of the standard systems of units. In MKS units length is given in meters (m), time in seconds (sec), mass in kilograms (kg), and force in Newtons (1 Newton = 1 kg m/sec^2). In CGS units length is given in centimeters (cm), time in seconds (sec), mass in grams (g), and force in dynes (1 dyne = 1 g cm/sec^2). In British units length is given in feet (ft), time in seconds (sec), mass in slugs (sl), and force in pounds (1 lb = 1 sl ft/sec^2).

The gravitational force F_{grav} is simply the downward weight of the mass. If we assume a uniform gravitational acceleration g then

$$F_{grav} = -mg\,, \tag{7.2}$$

where $g = 9.8$ m/sec^2 in MKS units, $g = 980$ cm/sec^2 in CGS units, and $g = 32$ ft/sec^2 in British units.

The spring force is modeled by Hooke's law

$$F_{spr} = -k(y - y_o)\,, \tag{7.3}$$

where k is the so-called spring constant or spring coefficient. This is a fairly good model provided $y - y_o$ does not get too big. When there is no external driving force, the mass has a rest position $y_r < y_o$ that satisfies

$$0 = F_{grav} + F_{spr} \quad \text{at } y = y_r\,.$$

Hence, we have

$$mg = -k(y_r - y_o) = k(y_o - y_r) = k|y_r - y_o|\,. \tag{7.4}$$

Sometimes you will be given $|y_r - y_o|$ and have to figure out k from this relation.

The damping force is modeled by

$$F_{damp} = -\gamma \frac{dy}{dt}, \tag{7.5}$$

where $\gamma \geq 0$ is the so-called damping coefficient. This is not as good a model for damping force as Hooke's Law was for the spring force, but we will use it because of its simplicity. Sometimes you will be given $|F_{damp}|$ at a particular speed and have to determine γ from this relation.

If we place (7.2), (7.3), and (7.5) into Newton's law of motion (7.1) and neglect the external driving, we obtain

$$m\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + ky = ky_o - mg\,.$$

We see from (7.4) that $ky_o - mg = ky_r$, where y_r is the rest position. We thereby have

$$m\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + ky = ky_r\,.$$

This clearly has the particular solution $y = y_r$. If we let $y(t) = y_r + h(t)$ then $h(t)$ satisfies the homogeneous equation

$$m\frac{d^2h}{dt^2} + \gamma \frac{dh}{dt} + kh = 0\,.$$

Here $h(t)$ is simply the *displacement* of the mass from its rest position y_r. If the external driving is present, this becomes

$$m\frac{d^2h}{dt^2} + \gamma \frac{dh}{dt} + kh = F_{ext}(t)\,. \tag{7.6}$$

We will study the motion of this spring-mass system building up its complexity from simplest case.

7.2. Unforced, Undamped Motion ($F_{ext} = 0$, $\gamma = 0$). In this case (7.6) reduces to

$$m\frac{d^2h}{dt^2} + kh = 0\,,$$

or in normal form

$$\frac{d^2h}{dt^2} + \frac{k}{m}h = 0\,. \tag{7.7}$$

Its characteristic polynomial is

$$p(z) = z^2 + \frac{k}{m}\,,$$

which has roots $\pm i\omega_o$ where

$$\omega_o = \sqrt{\frac{k}{m}} \,. \tag{7.8}$$

A general solution of equation (7.7) is

$$h(t) = c_1 \cos(\omega_o t) + c_2 \sin(\omega_o t) \,. \tag{7.9}$$

For the initial condtions $h(0) = h_0$ and $h'(0) = h_1$ this becomes

$$h(t) = h_0 \cos(\omega_o t) + h_1 \frac{\sin(\omega_o t)}{\omega_o} \,.$$

Such motion is called *simple harmonic motion.* It involves the single frequency ω_o.

Because ω_o is associated with the spring constant k through (7.8), it is called the *natural frequency* of the spring. Therefore the associated *natural period* T_o is

$$T_o = \frac{2\pi}{\omega_o} \,.$$

In MKS, CGS, and British units ω_o is given in radians/sec, or simply 1/sec because radians are considered to be nondimensional. Then T_o is given in sec.

The simple harmonic motion (7.9) is nontrivial whenever either c_1 or c_2 is nonzero. In that case we can express it in the so-called applitude-phase form

$$h(t) = A \cos(\omega_o t - \delta) \,,$$

where $A > 0$ is its amplitude and δ in $[0, 2\pi)$ is its phase. By the cosine addition formula the above form can be expanded as

$$h(t) = A \cos(\delta) \cos(\omega_o t) + A \sin(\delta) \sin(\omega_o t) \,.$$

Upon comparing this with (7.9) we see that

$$A \cos(\delta) = c_1 \,, \qquad A \sin(\delta) = c_2 \,.$$

This shows that (A, δ) are simply the polar coordinates of the point in the plane whose cartesian coordinates are (c_1, c_2). Clearly $A = \sqrt{c_1^2 + c_2^2} > 0$ while δ satisfies

$$\cos(\delta) = \frac{c_1}{A} \,, \qquad \sin(\delta) = \frac{c_2}{A} \,.$$

There is a unique δ in $[0, 2\pi)$ that satisfies these equations.

Example. A mass of 10 grams stretches a spring 5 cm when at rest. At $t = 0$ the mass is set in motion from its rest position with a downward velocity of 35 cm/sec. Neglect damping and external forces.

a) What is the displacement of the mass as a function of time?
b) What is the amplitude, phase, frequency, and period of the motion?
c) At what positive time does the mass first return to its rest position?

Solution. Because $g = 980$ cm/sec^2, we can find k by setting

$$k \cdot 5 = mg = 10 \cdot 980 \quad \text{dynes}\,,$$

whereby

$$k = \frac{10 \cdot 980}{5} \quad \text{dynes/cm}\,.$$

Because we are neglecting damping and external forces, the equation of motion takes the form

$$m\,\frac{d^2h}{dt^2} + k\,h = 0\,,$$

which becomes

$$10\,\frac{d^2h}{dt^2} + \frac{10 \cdot 980}{5}\,h = 0\,.$$

Bringing this into normal form gives

$$\frac{d^2h}{dt^2} + \frac{980}{5}\,h = 0\,,$$

which becomes

$$\frac{d^2h}{dt^2} + 196\,h = 0\,.$$

Because $\omega_o^2 = 196$, one sees that $\omega_o = 14$ 1/sec.

Therefore a general solution of the equation of motion is

$$h(t) = c_1 \cos(14t) + c_2 \sin(14t)\,.$$

The initial conditions are $h(0) = 0$ and $h'(0) = -35$ cm/sec. Because

$$h'(t) = -14c_1 \sin(14t) + 14c_2 \cos(14t)\,,$$

the boundary conditions imply that

$$h(0) = c_1 = 0\,, \qquad h'(0) = 14c_2 = -35\,,$$

which implies $c_1 = 0$ and $c_2 = -\frac{5}{2}$. From this we can read off the following.

a) The displacement of the mass as a function of time is

$$h(t) = -\tfrac{5}{2}\sin(14t) = \tfrac{5}{2}\cos(14t - \tfrac{3\pi}{2}) \quad \text{cm}\,.$$

b) The amplitude of the motion is $\frac{5}{2}$ cm, the phase is $\frac{3\pi}{2}$, the frequency is 14 1/sec, and the period is $\frac{\pi}{7}$ sec.

c) The positive time at which the mass first returns to its rest position is $t = \frac{\pi}{14}$.

7.3. Unforced, Damped Motion ($F_{ext} = 0$, $\gamma > 0$). In this case (7.6) reduces to

$$m\frac{d^2h}{dt^2} + \gamma\frac{dh}{dt} + kh = 0\,,$$

which has the normal form

$$\frac{d^2h}{dt^2} + \frac{\gamma}{m}\frac{dh}{dt} + \frac{k}{m}h = 0\,. \tag{7.10}$$

Its characteristic polynomial is

$$p(z) = z^2 + \frac{\gamma}{m}z + \frac{k}{m}\,.$$

If we complete the square this has the form

$$p(z) = (z+\mu)^2 + \omega_o^2 - \mu^2\,. \tag{7.11}$$

where the damping rate μ and the natural frequency ω_o are defined by

$$\mu = \frac{\gamma}{2m}\,, \qquad \omega_o = \sqrt{\frac{k}{m}}\,.$$

It is clear there are three cases to consider.

- When $0 < \mu < \omega_o$ there is a conjugate pair of roots $-\mu \pm i\nu$ where

$$\nu = \sqrt{\omega_o^2 - \mu^2}\,. \tag{7.12}$$

- When $\mu = \omega_o$ there is a real double root $-\mu$, $-\mu$.
- When $\mu > \omega_o$ there is two simple real roots $-\mu \pm \sqrt{\mu^2 - \omega_o^2}$.

These are called the under damped, critically damped, and over damped cases respectively. Notice that you do not have to memorize any formulas here because you can simply read off which case a system is in from the roots of its characteristic polynomial: a conjugate pair of roots means that the system is under damped; a double real root means that the system is critically damped; two simple real roots means that the system is over damped.

For the *under damped* case a general solution is

$$h(t) = c_1 e^{-\mu t}\cos(\nu t) + c_2 e^{-\mu t}\sin(\nu t)\,. \tag{7.13}$$

Whenever either c_1 or c_2 is nonzero this can be put into the amplitude-phase form

$$h(t) = Ae^{-\mu t}\cos(\nu t - \delta)\,,$$

where $A = \sqrt{c_1^2 + c_2^2} > 0$ and $0 \le \delta < 2\pi$ satisfies

$$\cos(\delta) = \frac{c_1}{A}\,, \qquad \sin(\delta) = \frac{c_2}{A}\,.$$

Therefore the displacement is an exponentially decaying simple harmonic motion with the time-dependent amplitude $Ae^{-\mu t}$, frequency ν, and phase δ. In this context ν given by (7.12) is called the *quasi frequency* of the system and the associated period $2\pi/\nu$ is called the *quasi period*. Notice that

$$\nu < \omega_o\,, \qquad \frac{2\pi}{\nu} > T_o\,.$$

In other words, the quasi frequency is always less than the natural frequency, while the quasi period is always greater than the natural period.

For the *critically damped* case a general solution is

$$h(t) = c_1 e^{-\mu t} + c_2 t\, e^{-\mu t}\,. \tag{7.14}$$

Therefore the displacement has at most one zero and decays like $t\,e^{-\mu t}$ whenever $c_2 \neq 0$.

For the *over damped* case a general solution is

$$h(t) = c_1 e^{-\mu_+ t} + c_2 e^{-\mu_- t}\,, \tag{7.15}$$

where μ_+ and μ_- are defined by

$$\mu_\pm = \mu \pm \sqrt{\mu^2 - \omega_o^2}\,. \tag{7.16}$$

The subscript $+$ or $-$ simply indicates the sign of the square root taken in the above formula. Notice that $0 < \mu_- < \mu < \mu_+$. Therefore the displacement has at most one zero and decays like $e^{-\mu_- t}$ whenever $c_2 \neq 0$. Because $\mu_- < \mu$ one sees that in this case the decay of the displacement is slower than in either the under or critically damped cases.

Remark. The spring system is said to be extremely over damped when μ is much greater than ω_o. In that case we can use the approximation

$$\sqrt{\mu^2 - \omega_o^2} = \mu\sqrt{1 - \frac{\omega_o^2}{\mu^2}} \approx \mu\left(1 - \frac{\omega_o^2}{2\mu^2}\right) = \mu - \frac{\omega_o^2}{2\mu}\,,$$

to approximate μ_- and μ_+ in (7.16) by

$$\mu_- \approx \frac{\omega_o^2}{2\mu}\,, \qquad \mu_+ \approx 2\mu - \frac{\omega_o^2}{2\mu}\,.$$

In this regime these decay rates are very different from each other with

$$\frac{\mu_-}{\mu_+} \approx \frac{\omega_o^2}{4\mu^2}\,, \qquad \text{which is much less than } 1\,.$$

Remark. This damped spring system is a good model for shock absorbers in a car. When the shock absorbers are over damped one gets a jarring ride, while when they are under damped one gets a bouncy ride. Shock absorbers are tuned to be critically damped, which gives the least jarring and least bouncy ride.

7.4. Forced, Undamped Motion ($F_{ext} \neq 0$, $\gamma = 0$). In this case (7.6) reduces to

$$m\frac{d^2h}{dt^2} + kh = F_{ext}(t) \,.$$

We will study external forces of the form

$$F_{ext}(t) = F\cos(\omega t) \,.$$

The equation then has the normal form

$$\frac{d^2h}{dt^2} + \omega_o^2 h = a\cos(\omega t) \,, \tag{7.17}$$

where the natural frequency ω_o and the driving acceleration a are given by

$$\omega_o = \sqrt{\frac{k}{m}} \,, \qquad a = \frac{F}{m} \,.$$

Equation (7.17) may be solved either by Undetermined Coefficients or by Key Identity Evaluations. The characteristic polynomial is $p(z) = z^2 + \omega_o^2$, which has roots $\pm i\omega_o$. The forcing has characteristic $\pm i\omega$, degree $d = 0$, multiplicity $m = 0$ when $\omega \neq \omega_o$, and multiplicity $m = 1$ when $\omega = \omega_o \neq 0$.

For $\omega \neq \omega_o$, if we impose the inital conditions

$$h(0) = h_0 \,, \quad \text{and} \quad h'(0) = h_1 \,,$$

then the solution is found to be

$$h(t;\omega) = h_0\cos(\omega_o t) + h_1\frac{\sin(\omega_o t)}{\omega_o} + a\,\frac{\cos(\omega t) - \cos(\omega_o t)}{\omega_o^2 - \omega^2} \,. \tag{7.18}$$

This is not simple harmonic motion because two frequencies are involved. Such motion is called biharmonic. In general, whenever more than one frequency is involved the motion is called polyharmonic.

For $\omega = \omega_o \neq 0$, if we impose the inital conditions

$$h(0) = h_0 \,, \quad \text{and} \quad h'(0) = h_1 \,,$$

then the solution is found to be

$$h(t;\omega_o) = h_0\cos(\omega_o t) + h_1\frac{\sin(\omega_o t)}{\omega_o} + a\,\frac{t\,\sin(\omega_o t)}{2\omega_o} \,. \tag{7.19}$$

This is also not simple harmonic motion. In fact, its amplitude grows linearly in t! This phenomenon of *resonance* that occurs when the driving frequency ω becomes equal to the natural frequency ω_o of the system. Because the l'Hopital rule implies

$$\lim_{\omega\to\omega_o} \frac{\cos(\omega t) - \cos(\omega_o t)}{\omega_o^2 - \omega^2} = \lim_{\omega\to\omega_o} \frac{-t\,\sin(\omega t)}{-2\omega} = \frac{t\,\sin(\omega_o t)}{2\omega_o} \,,$$

we see that formula (7.19) is what we obtain by taking the limit $\omega \to \omega_o$ in formula (7.18).

We can understand the onset of resonance as $\omega \to \omega_o$ by using the identity

$$\cos(\omega t) - \cos(\omega_o t) = -2\sin\left(\frac{\omega - \omega_o}{2}\, t\right) \sin\left(\frac{\omega + \omega_o}{2}\, t\right),$$

to re-express formula (7.18) as

$$h(t;\omega) = h_0 \cos(\omega_o t) + h_1 \frac{\sin(\omega_o t)}{\omega_o} + A_\omega(t)\, \sin\left(\frac{\omega + \omega_o}{2}\, t\right),$$

where

$$A_\omega(t) = \frac{2a}{\omega^2 - \omega_o^2}\, \sin\left(\frac{\omega - \omega_o}{2}\, t\right).$$

When $\omega - \omega_o$ is very small compared to ω and ω_o then $A(t)$ will be a very slowly varying function of t compared to $\sin\big((\omega+\omega_o)t/2\big)$. In that case $\sin\big((\omega+\omega_o)t/2\big)$ will oscillate very many times during a period over which $A(t)$ oscillates just once. These rapid oscillations will have an amplitude of $|A(t)|$, which slowly oscillates between 0 and $2a/(\omega^2 - \omega_o^2)$. This slow oscillation is the phenomenon of *beating*. The so-called *beating frequency* is $\omega - \omega_o$, which is small, while the so-called *beating period* is $2\pi/(\omega - \omega_o)$, which is large. As $\omega \to \omega_o$ the beating frequency vanishes, the beating period diverges to infinity, while by the l'Hospital rule we see that

$$\lim_{\omega \to \omega_o} A_\omega(t) = \lim_{\omega \to \omega_o} \frac{2a \sin\left(\frac{\omega - \omega_o}{2}\, t\right)}{\omega^2 - \omega_o^2} = \lim_{\omega \to \omega_o} \frac{a \cos\left(\frac{\omega - \omega_o}{2}\, t\right) t}{2\omega} = \frac{a\, t}{2\omega_o}.$$

This is in accord with the amplitude we found in formula (7.19).

7.5. Forced, Damped Motion ($F_{ext} \neq 0$, $\gamma > 0$). In this case (7.6) reduces to

$$m \frac{d^2 h}{dt^2} + \gamma \frac{dh}{dt} + kh = F_{ext}(t).$$

We will again study external forces of the form

$$F_{ext}(t) = F \cos(\omega t).$$

The equation then has the normal form

$$\frac{d^2 h}{dt^2} + \frac{\gamma}{m}\frac{dh}{dt} + \frac{k}{m}\, h = \frac{F}{m}\, \cos(\omega t). \tag{7.20}$$

The associated homogeneous equation is (7.10), which descibes the associated unforced, damped system.

Once again we introduce the damping rate μ, the natural frequency ω_o, and the driving acceleration a defined by

$$\mu = \frac{\gamma}{2m}\,, \qquad \omega_o = \sqrt{\frac{k}{m}}\,, \qquad a = \frac{F}{m}\,. \tag{7.21}$$

The solution of the associated homogeneous equation $h_H(t)$ is then given by either (7.13), (7.14), or (7.15) depending on whether the associated unforced system is under damped, critically damped, or over damped. In all of these cases $h_H(t)$ decays to zero as $t \to \infty$.

A particular solution $h_P(t)$ of (7.20) can be found either by Undetermined Coefficients or by Key Identity Evaluations. The forcing has characteristic $i\omega$, degree $d = 0$, and multiplicity $m = 0$. We find that

$$h_P(t) = \frac{a(\omega_o^2 - \omega^2)}{(\omega_o^2 - \omega^2)^2 + 4\mu^2\omega^2}\,\cos(\omega t) + \frac{a2\mu\omega}{(\omega_o^2 - \omega^2)^2 + 4\mu^2\omega^2}\,\sin(\omega t)\,,$$

where μ, ω_o, and a are given by (7.21). This is simple harmonic motion that can be put into the amplitude-phase form

$$h_P(t) = A\cos(\omega t - \delta)\,, \tag{7.22}$$

where

$$A = \frac{a}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\mu^2\omega^2}}\,, \qquad \delta = \cos^{-1}\left(\frac{\omega_o^2 - \omega^2}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\mu^2\omega^2}}\right).$$

Because it is periodic, this particular solution is called the *steady solution* of the forced, damped system. A general solution of this system thereby has the form

$$h(t) = h_H(t) + h_P(t)\,,$$

where $h_H(t)$ decays to zero as $t \to \infty$ and $h_P(t)$ is the steady solution given by (7.22). For this reason $h_H(t)$ is called the *transient component* of the solution, or simply the *transient*.

Finally, notice that the resonance phenomenon is modified by the presence of damping. In particular, the solutions will remain bounded for any driving frequency ω. It is clear from (7.22) that the amplitude of the steady solution will be maximum when $(\omega_o^2 - \omega^2)^2 + 4\mu^2\omega^2$ is minimum. When $4\mu^2 < \omega_o^2$ then this happens when

$$\omega = \sqrt{\omega_o^2 - 4\mu^2}\,,$$

while when $4\mu^2 \geq \omega_o^2$ then this happens when $\omega = 0$. In either case it happens when ω is less than the natural frequency.