

**FIRST-ORDER SYSTEMS OF
ORDINARY DIFFERENTIAL EQUATIONS III:
Autonomous Planar Systems**

David Levermore
Department of Mathematics
University of Maryland

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Because the presentation of this material in lecture will differ from that in the book, I felt that notes that closely follow the lecture presentation might be appreciated.

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6. INTEGRAL METHODS

For the remainder of the course we will study first-order, autonomous, planar systems in the normal form

$$(6.1) \quad \frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y).$$

The word “autonomous” has Greek roots and means “self governing”. Such systems are called *autonomous* because they only depend on the state (x, y) and not on time t . What makes them *planar* is the fact they have two dependent variables, x and y .

In the general discussions throughout this section we will assume that f and g are continuously differentiable over some domain D in the xy -plane. Therefore our basic existence and uniqueness theorem for initial-value problems applies whenever the initial data lies in D . When D is not specified explicitly, assume it is the entire xy -plane.

6.1. Stationary Solutions. A solution of system (6.1) called *stationary* if it does not depend in time. Because the time derivatives are zero for such a solution, it must satisfy the algebraic system

$$(6.2) \quad 0 = f(x, y), \quad 0 = g(x, y).$$

Conversely, if (x_o, y_o) satisfies this algebraic system then it is easily checked that $(x(t), y(t)) = (x_o, y_o)$ is a stationary solution of system (6.1). Therefore we can find all stationary solutions of the first-order system (6.1) by finding all solutions of the algebraic system (6.2).

System (6.2) consists of two algebraic equations for the two unknowns x and y . However, in general this system can have no solutions, one solution, or many solutions.

Example. Find all stationary solutions of the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3.$$

Solution. The stationary solutions satisfy

$$0 = y, \quad 0 = x - x^3 = x(1 - x)(1 + x).$$

The first of these equations implies $y = 0$, while the second implies $x = 0$, $x = 1$, or $x = -1$. Therefore the stationary solutions are

$$(0, 0), \quad (1, 0), \quad (-1, 0).$$

Example. Find all stationary solutions of the system

$$\frac{dx}{dt} = (y - x)(x - 1), \quad \frac{dy}{dt} = (3 + 2x - x^2)y.$$

Solution. The stationary solutions satisfy

$$0 = (y - x)(x - 1), \quad 0 = (3 + 2x - x^2)y = (3 - x)(1 + x)y.$$

The first of these equations implies $y = x$ or $x = 1$. Each of these possibilities spawns a logical thread that must be followed.

First, if $y = x$ then the second equation becomes $0 = (3 - x)(1 + x)x$, which implies $x = 3$, $x = -1$, or $x = 0$. Hence, the thread from $y = x$ leads to the stationary solutions $(3, 3)$, $(-1, -1)$, and $(0, 0)$.

On the other hand, if $x = 1$ then the second equation becomes $0 = 4y$, which implies $y = 0$. Hence, the thread from $x = 1$ leads to the stationary solution $(1, 0)$.

By collecting the results from each thread, we have found the four stationary solutions

$$(3, 3), \quad (-1, -1), \quad (0, 0), \quad (1, 0).$$

Remark. We could have found the stationary solutions in the last example by starting with the second equation. That equation implies that $x = 3$, $x = -1$, or $y = 0$. Each of these possibilities spawns a logical thread that must be followed.

6.2. Semistationary Solutions. A solution of system (6.1) called *semistationary* if one of its entries does not vary with time. These special solutions of system (6.1) are easy to identify. They can be analyzed by the methods for first-order equations we studied at the beginning of the course.

There will be a semistationary solution in the form $(a, Y(t))$ if the right-hand side of the dx/dt equation vanishes when $x = a$ for every y — i.e. if $f(a, y) = 0$ for every y . In that case $y = Y(t)$ must satisfy the first-order equation

$$\frac{dy}{dt} = g(a, y).$$

Sometimes we can find explicit or implicit solutions of this first-order equation. We can always understand the behavior of these solutions with a phase-line portrait.

There will be a semistationary solution in the form $(X(t), b)$ if the right-hand side of the dy/dt equation vanishes when $y = b$ for every x — i.e. if $g(x, b) = 0$ for every x . In that case $x = X(t)$ must satisfy the first-order equation

$$\frac{dx}{dt} = f(x, b).$$

Sometimes we can find explicit or implicit solutions of this first-order equation. We can always understand the behavior of these solutions with a phase-line portrait.

Example. Find all semistationary solutions of the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3.$$

Solution. There is no value of x that makes the right-hand side of the dx/dt equation zero. There is no value of y that makes the right-hand side of the dy/dt equation zero. Therefore this system has no semistationary solutions.

Example. Find all semistationary solutions of the system

$$\frac{dx}{dt} = (y - x)(x - 1), \quad \frac{dy}{dt} = (3 + 2x - x^2)y.$$

Solution. When $x = 1$ the right-hand side of the dx/dt equation is zero. When $x = 1$ the dy/dt equation reduces to the first-order equation

$$\frac{dy}{dt} = (3 + 2 \cdot 1 - 1^2)y = 4y.$$

The solution of this equation with initial-value y_o at $t = 0$ is $y = y_o e^{4t}$.

When $y = 0$ the right-hand side of the dy/dt equation is zero. When $y = 0$ the dx/dt equation reduces to the first-order equation

$$\frac{dx}{dt} = (0 - x)(x - 1) = x(1 - x).$$

The solution of this equation with initial-value x_o at $t = 0$ is $x = x_o e^t / (1 + x_o(e^t - 1))$.

Therefore this system has the semistationary solutions

$$(1, y_o e^{4t}), \quad \left(\frac{x_o e^t}{1 + x_o(e^t - 1)}, 0 \right).$$

Remark. For $y_o = 0$, $x_o = 0$, and $x_o = 1$ these recover the stationary solutions $(1, 0)$, $(0, 0)$, and $(1, 0)$ respectively. We thereby see that the stationary point $(1, 0)$ is included in both semistationary solutions, the stationary point $(0, 0)$ is included in one semistationary solution, while the stationary points $(3, 3)$ and $(-1, -1)$ are not included in any semistationary solution.

6.3. Reduction to First-Order Equations. Sometimes system (6.1) can be solved analytically by solving two first-order equations. The first of these is the so-called *orbit equation*. It seeks y as a function of x that satisfies

$$(6.3) \quad \frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}.$$

This first-order equation can be solved if it is linear, is separable, or has an exact differential form. If we can find an explicit solution $y = Y(x)$ of the orbit equation then we can try to find x as a function of t by solving the so-called *reduced equation*,

$$(6.4) \quad \frac{dx}{dt} = f(x, Y(x)).$$

This first-order equation is autonomous, so it can be solved implicitly if we can find a primitive of $1/f(x, Y(x))$. If we are able to find a solution $x(t)$ of the reduced equation then a solution of system (6.1) is given by

$$(6.5) \quad (x(t), y(t)) = (x(t), Y(x(t))).$$

It is often difficult or impossible to solve the orbit equation (6.3). And even when we can find an explicit solution $Y(x)$ of the orbit equation, it is often difficult or impossible to find a primitive of $1/f(x, Y(x))$ in order to solve the reduced equation (6.4).

Example. Try to solve the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3.$$

Solution. The orbit equation is

$$\frac{dy}{dx} = \frac{x - x^3}{y},$$

which is separable. It has the separated form

$$y \, dy = (x - x^3) \, dx,$$

which can be integrated to obtain

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{2}c.$$

Here we put the $\frac{1}{2}$ in front of the c because we see that the equation must be multiplied by 2 in order to solve for y . Upon solving for y we find the explicit solutions

$$y = \pm \sqrt{x^2 - \frac{1}{2}x^4 + c}.$$

The reduced equation then becomes

$$\frac{dx}{dt} = \pm \sqrt{x^2 - \frac{1}{2}x^4 + c},$$

which can be solved implicitly in terms of an integral as

$$t = \pm \int \frac{1}{\sqrt{x^2 - \frac{1}{2}x^4 + c}} \, dx.$$

This integral cannot be evaluated analytically for every c by methods from first-year calculus. That evaluation requires the use of *elliptic functions*, which lie beyond the scope of this course. It can be evaluated analytically for $c = 0$ by methods from first-year calculus, but we will not do that here.

Remark. In the previous example we could integrate the orbit equation easily, but could not do the same for the reduced equation. This is often the case when the orbit equation can be integrated. In the next example we cannot even integrate the orbit equation. Most orbit equations cannot be integrated.

Example. Try to solve the system

$$\frac{dx}{dt} = (y - x)(x - 1), \quad \frac{dy}{dt} = (3 + 2x - x^2)y.$$

Solution. The orbit equation is

$$\frac{dy}{dx} = \frac{(3 + 2x - x^2)y}{(y - x)(x - 1)}.$$

This equation is not linear or separable. Later we will show that it cannot be integrated.

6.4. Hamiltonian Systems. The orbit equation (6.3) can be put into the differential form

$$-g(x, y) dx + f(x, y) dy = 0.$$

This differential form is exact when

$$(6.6) \quad \partial_x f(x, y) + \partial_y g(x, y) = 0.$$

In that case there exists a function $H(x, y)$ such that $\partial_x H(x, y) = -g(x, y)$ and $\partial_y H(x, y) = f(x, y)$.

$$\partial_x H(x, y) = -g(x, y), \quad \partial_y H(x, y) = f(x, y).$$

System (6.1) thereby is seen to have the form

$$(6.7) \quad \frac{dx}{dt} = \partial_y H(x, y), \quad \frac{dy}{dt} = -\partial_x H(x, y).$$

Such a system is said to be a *Hamiltonian* system while H is called its *Hamiltonian*. The Hamiltonian H is determined uniquely up to an additive constant. Because we have assumed that f and g are continuously differentiable, H will be twice continuously differentiable.

Example. Consider the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3.$$

Show it is Hamiltonian and find its Hamiltonian $H(x, y)$.

Solution. The system is Hamiltonian because

$$\partial_x f(x, y) + \partial_y g(x, y) = \partial_x y + \partial_y (x - x^3) = 0.$$

We can find $H(x, y)$ by solving

$$\partial_y H(x, y) = y, \quad \partial_x H(x, y) = -x + x^3.$$

Upon integrating the first equation we find that $H(x, y) = \frac{1}{2}y^2 + h(x)$. By substituting this into the second equation we find that $h'(x) = -x + x^3$. By setting $h(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ we obtain the Hamiltonian

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

Many systems can be put into Hamiltonian form. For example, every second-order equation of the form

$$\frac{d^2x}{dt^2} + g(x) = 0,$$

can be recast as the first-order system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -g(x).$$

This system is Hamiltonian with $H(x, y) = \frac{1}{2}y^2 + G(x)$ where $G'(x) = g(x)$ because

$$\partial_y H(x, y) = y, \quad \partial_x H(x, y) = G'(x) = g(x).$$

More generally, consider any first-order system of the form

$$\frac{dx}{dt} = f(y), \quad \frac{dy}{dt} = -g(x).$$

This system is Hamiltonian with $H(x, y) = F(y) + G(x)$ where $F'(y) = f(y)$ and $G'(x) = g(x)$ because

$$\partial_y H(x, y) = F'(y) = f(y), \quad \partial_x H(x, y) = G'(x) = g(x).$$

Remark. We can determine the global phase-plane portrait of a Hamiltonian system by analyzing the Hamiltonian $H(x, y)$. We will postpone the discussion of how this is done until after we have introduced a more larger class of systems to which the same techniques can be applied.

6.5. Conservative Systems. More generally, the orbit equation (6.3) can be put into the differential form

$$-\mu(x, y) g(x, y) dx + \mu(x, y) f(x, y) dy = 0,$$

where $\mu(x, y)$ is a positive factor that we will assume is also continuously differentiable. We hope to find a $\mu(x, y)$ that makes this differential form exact, in which case it is called an integrating factor. This will be the case when $\mu(x, y)$ satisfies

$$(6.8) \quad \partial_x [f(x, y) \mu] + \partial_y [g(x, y) \mu] = 0.$$

When we can find such a μ then there exists a function $H(x, y)$ such that

$$\partial_x H(x, y) = -\mu(x, y) g(x, y), \quad \partial_y H(x, y) = \mu(x, y) f(x, y).$$

System (6.1) thereby is seen to have the form

$$(6.9) \quad \frac{dx}{dt} = \frac{1}{\mu(x, y)} \partial_y H(x, y), \quad \frac{dy}{dt} = -\frac{1}{\mu(x, y)} \partial_x H(x, y).$$

Such a system is said to be *conservative*. Hamiltonian systems correspond to the cases when $\mu(x, y) = 1$. In the more general setting, H is called an *integral* of the system. Given a choice of μ , the integral H is determined uniquely up to an additive constant. Because we have assumed that f , g , and μ are continuously differentiable, H will be twice continuously differentiable.

Example. The drag force \mathbf{F}_{drag} acting on an object moving with velocity \mathbf{u} through air can be modeled as

$$\mathbf{F}_{\text{drag}} = -\rho_{\text{air}} A |\mathbf{u}| \mathbf{u},$$

where ρ_{air} is the density of air, A is the areodynamic cross-section of the object, and $|\mathbf{u}|$ is the magnitude of \mathbf{u} . This force has magnitude proportional to $|\mathbf{u}|^2$ and acts in the direction opposite to \mathbf{u} . If the mass of the object is m and the gravitational field is uniform then the velocity \mathbf{u} will evolve as

$$\frac{du}{dt} = -ku\sqrt{u^2 + v^2}, \quad \frac{dv}{dt} = -kv\sqrt{u^2 + v^2} - a,$$

where u and v are the horizontal and vertical components of \mathbf{u} , $a > 0$ is the magnitude of the gravitational acceleration, and k is given by $k = \rho_{\text{air}} A/m$. Find an integral for this system and use it to give an implicit general solution.

Solution. The only stationary solution of this system is

$$u = 0, \quad v = -\sqrt{a/k}.$$

The only semistationary solutions of this system have the form $(0, V(t))$ where $v = V(t)$ satisfies the first-order equation

$$\frac{dv}{dt} = -k|v|v - a.$$

We solved this equation earlier in the course.

We now seek solutions with $u \neq 0$. The system is not separable. Setting

$$f(u, v) = -ku\sqrt{u^2 + v^2}, \quad g(u, v) = -kv\sqrt{u^2 + v^2} - a,$$

we see that

$$\partial_u f = -k\sqrt{u^2 + v^2} - \frac{ku^2}{\sqrt{u^2 + v^2}}, \quad \partial_v g = -k\sqrt{u^2 + v^2} - \frac{kv^2}{\sqrt{u^2 + v^2}}.$$

The system is not Hamiltonian because

$$\partial_u f + \partial_v g = -3k\sqrt{u^2 + v^2} \neq 0.$$

Therefore we must seek an integrating factor.

An integrating factor $\mu(u, v)$ must satisfy the linear partial differential equation

$$\begin{aligned} 0 &= \partial_u[f\mu] + \partial_v[g\mu] = f\partial_u\mu + g\partial_v\mu + (\partial_u f + \partial_v g)\mu \\ &= -ku\sqrt{u^2 + v^2}\partial_u\mu - (kv\sqrt{u^2 + v^2} + a)\partial_v\mu - 3k\sqrt{u^2 + v^2}\mu. \end{aligned}$$

By setting $\partial_v\mu = 0$ this reduces to the linear ordinary differential equation

$$0 = u\partial_u\mu + 3\mu.$$

This has the nonzero solution $\mu = 1/u^3$ for $u \neq 0$.

Because $\mu = 1/u^3$ is an integrating factor wherever $u \neq 0$, we can find an integral $H(u, v)$ such that

$$\partial_u H(u, v) = \frac{1}{u^3} (kv\sqrt{u^2 + v^2} + a), \quad \partial_v H(u, v) = -\frac{1}{u^3} ku\sqrt{u^2 + v^2}.$$

We can integrate the second equation as

$$H(u, v) = -\int \frac{k}{u^2} \sqrt{u^2 + v^2} dv = -k \int \sqrt{1 + r^2} dr, \quad \text{where } v = ur.$$

By using the calculus fact (which you are not expected to memorize) that

$$\int \sqrt{1 + r^2} dr = \frac{1}{2} \left(r\sqrt{1 + r^2} + \log\left(r + \sqrt{1 + r^2}\right) \right) + c,$$

we obtain

$$H(u, v) = -\frac{k}{2} \left(\frac{v}{u} \sqrt{1 + \frac{v^2}{u^2}} + \log\left(\frac{v}{u} + \sqrt{1 + \frac{v^2}{u^2}}\right) \right) + h(u).$$

But, by using the same calculus fact, this implies that

$$\partial_u H(u, v) = -k\sqrt{1 + \frac{v^2}{u^2}} \partial_u \left(\frac{v}{u} \right) + h'(u) = k\sqrt{u^2 + v^2} \frac{v}{u^3} + h'(u),$$

which when placed into the first equation yields

$$h'(u) = \frac{a}{u^3}.$$

By taking $h(u) = -\frac{1}{2}a/u^2$ we obtain the integral

$$H(u, v) = -\frac{k}{2} \left(\frac{v}{u} \sqrt{1 + \frac{v^2}{u^2}} + \log \left(\frac{v}{u} + \sqrt{1 + \frac{v^2}{u^2}} \right) \right) - \frac{a}{2u^2}.$$

Therefore when $u \neq 0$ an implicit general solution is given by

$$k \left(\frac{v}{u} \sqrt{1 + \frac{v^2}{u^2}} + \log \left(\frac{v}{u} + \sqrt{1 + \frac{v^2}{u^2}} \right) \right) + \frac{a}{u^2} = c.$$

6.6. Phase-Plane Portraits of Conservative Systems. The global phase-plane portrait of a conservative system can be determined by analyzing its integral $H(x, y)$. This is due to the following facts.

Fact 1. A point (x_o, y_o) is a stationary solution of the conservative system (6.9) if and only if it is a critical point of $H(x, y)$.

Reason. Recall that a point is a *critical point* of $H(x, y)$ if and only if it satisfies

$$\partial_x H(x, y) = 0, \quad \partial_y H(x, y) = 0.$$

But these are exactly the equations that characterize stationary solutions of (6.9).

Fact 2. If $(x(t), y(t))$ is any solution to the conservative system (6.9) then $H(x(t), y(t))$ is a constant.

Reason. The multivariable chain rule and system (6.9) yield

$$\begin{aligned} \frac{d}{dt} H(x, y) &= \partial_x H(x, y) \frac{dx}{dt} + \partial_y H(x, y) \frac{dy}{dt} \\ &= \partial_x H(x, y) \frac{1}{\mu(x, y)} \partial_y H(x, y) + \partial_y H(x, y) \left(-\frac{1}{\mu(x, y)} \partial_x H(x, y) \right) = 0. \end{aligned}$$

Therefore $H(x(t), y(t))$ is a constant. □

This fact states that orbits of system (6.9) lie on so-called *level sets* of $H(x, y)$ — namely, sets in the xy -plane of the form

$$(6.10) \quad \{(x, y) : H(x, y) = c\} \quad \text{for some constant } c.$$

This set will be empty unless c is in the range of H . If c is in the range of H then c is called a *critical value* of H if $c = H(x_o, y_o)$ for some point (x_o, y_o) that is a critical point of the function H , and is called a *noncritical value* of H otherwise.

- If c is a noncritical value of H then the associated level set will consist of one or more disjoint curves, each of which is a single orbit. These curves will be either a closed loop, corresponding to a periodic orbit, or an unbounded curve, corresponding to an orbit that becomes unbounded as $t \rightarrow -\infty$ or as $t \rightarrow \infty$.
- If c is a critical value of H then the associated level set will consist of one or more critical points and possibly other curves that can either loop from a critical point to itself, connect two critical points, connect a critical point to infinity, run from infinity to infinity, or be a closed loop. We will illustrate most of these possibilities with examples below.

To simplify our discussion, we consider only functions H whose critical points are *nondegenerate*. Recall that a critical point (x_o, y_o) of H is said to be nondegenerate if

$$(6.11) \quad \det(\mathbf{H}(x_o, y_o)) \neq 0,$$

where $\mathbf{H}(x, y)$ denotes the Hessian matrix of second derivatives, which is given by

$$(6.12) \quad \mathbf{H}(x, y) = \begin{pmatrix} \partial_{xx}H(x, y) & \partial_{xy}H(x, y) \\ \partial_{yx}H(x, y) & \partial_{yy}H(x, y) \end{pmatrix}.$$

Because $\partial_{xy}H(x, y) = \partial_{yx}H(x, y)$, the Hessian matrix is symmetric. Therefore it has real eigenvalues. Recall that $\det(\mathbf{H}(x, y))$ will be the product of those eigenvalues. If (x_o, y_o) is a nondegenerate critical point of H then the eigenvalues of $\mathbf{H}(x_o, y_o)$ are nonzero and there are three possibilities:

- if $\mathbf{H}(x_o, y_o)$ has one eigenvalue of each sign then (x_o, y_o) is a saddle point of H ;
- if $\mathbf{H}(x_o, y_o)$ has two positive eigenvalues then (x_o, y_o) is a local minimizer of H ;
- if $\mathbf{H}(x_o, y_o)$ has two negative eigenvalues then (x_o, y_o) is a local maximizer of H .

In the first case the level sets of H near (x_o, y_o) moves away from (x_o, y_o) . In that case (x_o, y_o) is a *saddle* in the phase portrait of system (6.9). This last case arises whenever

$$\det(\mathbf{H}(x_o, y_o)) < 0.$$

In the last two cases the level sets of H near (x_o, y_o) will be closed loops, representing periodic orbits. In those cases (x_o, y_o) is a *center* in the phase portrait of system (6.9). One of these first two cases arises whenever

$$\det(\mathbf{H}(x_o, y_o)) > 0.$$

With the convention we have adopted here, if $\mu(x_o, y_o) > 0$ then (x_o, y_o) will be a clockwise center when (x_o, y_o) is a local minimizer of H and a counterclockwise center when (x_o, y_o) is a local maximizer of H .

Example. Draw a phase portrait for the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3.$$

Solution. We have shown already that this is a Hamiltonian system with

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

We also have shown already that the critical points of H are

$$(0, 0), \quad (-1, 0), \quad (1, 0).$$

The associated critical values are

$$H(0,0) = 0, \quad H(\pm 1,0) = -\frac{1}{4}.$$

The Hessian matrix is

$$\mathbf{H}(x,y) = \begin{pmatrix} -1 + 3x^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Because

$$\mathbf{H}(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{H}(\pm 1,0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

we see that $(0,0)$ is a *saddle point* while $(-1,0)$ and $(1,0)$ are *local minimizers*. Hence, the phase portrait is a *saddle* near $(0,0)$ and a *clockwise center* near $(-1,0)$ and $(1,0)$.

Because $H(x,y) \rightarrow \infty$ as $x^2 + y^2 \rightarrow \infty$ while $(-1,0)$ and $(1,0)$ are the only local minimizers and they have the same value $H(\pm 1,0) = -\frac{1}{4}$, these local minimizers are global minimizers. Hence, the level set associated with the critical value $-\frac{1}{4}$ consists of just the critical points $(-1,0)$ and $(1,0)$. This can also be seen from the fact the equation $H(x,y) = -\frac{1}{4}$ can be written as

$$\frac{1}{2}y^2 + \frac{1}{4}(x^2 - 1)^2 = 0,$$

which is satisfied only when $(x,y) = (\pm 1,0)$.

The level set associated with the critical value 0 consists of the saddle point $(0,0)$ plus all other points (x,y) that satisfy $H(x,y) = 0$. This equation can be written as

$$y^2 = x^2 - \frac{1}{2}x^4,$$

which has solutions

$$y = \pm \sqrt{x^2 - \frac{1}{2}x^4} \quad \text{for every } x \text{ in } [-\sqrt{2}, \sqrt{2}].$$

A graph shows two orbits that each emerge from the origin, moves out to either $(\pm\sqrt{2},0)$, and returns to the origin.

7. NONINTEGRAL METHODS

For most autonomous planar systems the orbit equation (2.3) cannot be integrated. In this section we present techniques for understanding the phase portrait of such a system that do not require finding an integral. We consider first-order, autonomous, planar systems that have the form

$$(7.1) \quad \frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y).$$

7.1. Linearization about Stationary Solutions. Recall from multivariable calculus that the Taylor approximations of $f(x, y)$ and $g(x, y)$ near a point (x_o, y_o) have the form

$$\begin{aligned} f(x, y) &= f(x_o, y_o) + (x - x_o) \partial_x f(x_o, y_o) + (y - y_o) \partial_y f(x_o, y_o) + \text{higher-order terms}, \\ g(x, y) &= g(x_o, y_o) + (x - x_o) \partial_x g(x_o, y_o) + (y - y_o) \partial_y g(x_o, y_o) + \text{higher-order terms}. \end{aligned}$$

If we drop the higher-order terms then this becomes a linear approximation to $f(x, y)$ and $g(x, y)$ that will be valid when (x, y) is near (x_o, y_o) .

If (x_o, y_o) is a stationary solution of our system then $f(x_o, y_o) = g(x_o, y_o) = 0$ and these linear approximations become

$$\begin{aligned} f(x, y) &\approx (x - x_o) \partial_x f(x_o, y_o) + (y - y_o) \partial_y f(x_o, y_o), \\ g(x, y) &\approx (x - x_o) \partial_x g(x_o, y_o) + (y - y_o) \partial_y g(x_o, y_o). \end{aligned}$$

The idea of linearization is that when the solution $(x(t), y(t))$ of system is near (x_o, y_o) then it can be approximated by the solution of a linear system obtained by replacing $f(x, y)$ and $g(x, y)$ with the above linear approximations. Specifically, the idea is that $(x(t), y(t)) \approx (x_o + \tilde{x}(t), y_o + \tilde{y}(t))$ where $(\tilde{x}(t), \tilde{y}(t))$ satisfy the linear system

$$(7.2) \quad \frac{d}{dt} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \partial_x f(x_o, y_o) & \partial_y f(x_o, y_o) \\ \partial_x g(x_o, y_o) & \partial_y g(x_o, y_o) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

This is called the *linearization of system (7.1) about (x_o, y_o)* . The coefficient matrix of this linearized system is $\mathbf{A} = \mathbf{J}(x_o, y_o)$ where $\mathbf{J}(x, y)$ is the matrix of partial derivatives given by

$$(7.3) \quad \mathbf{J}(x, y) = \begin{pmatrix} \partial_x f(x, y) & \partial_y f(x, y) \\ \partial_x g(x, y) & \partial_y g(x, y) \end{pmatrix}.$$

This matrix is called the *Jacobian matrix*. Notice that the partial derivatives of f are the entries of its top row while those of g are the entries of its bottom row.

Example. Compute the coefficient matrix of the linearization about each stationary point of the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3.$$

Solution. We have seen that the stationary points of this system are

$$(-1, 0), \quad (0, 0), \quad (1, 0).$$

The Jacobian matrix for the system is

$$\mathbf{J}(x, y) = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}.$$

At $(0, 0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \mathbf{J}(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

At $(-1, 0)$ and $(1, 0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \mathbf{J}(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ 1 - 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

Example. Compute the coefficient matrix of the linearization about each stationary point of the system

$$\frac{dx}{dt} = (y - x)(x - 1), \quad \frac{dy}{dt} = (3 + 2x - x^2)y.$$

Solution. We have seen that the stationary points of this system are

$$(-1, -1), \quad (0, 0), \quad (1, 0), \quad (3, 3).$$

The Jacobian matrix for the system is

$$\mathbf{J}(x, y) = \begin{pmatrix} y - 2x + 1 & x - 1 \\ 2y - 2xy & 3 + 2x - x^2 \end{pmatrix}.$$

At $(-1, -1)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \mathbf{J}(-1, -1) = \begin{pmatrix} -1 + 2 + 1 & -1 - 1 \\ -2 - 2 & 3 - 2 - 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -4 & 0 \end{pmatrix}.$$

At $(0, 0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \mathbf{J}(0, 0) = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}.$$

At $(1, 0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \mathbf{J}(1, 0) = \begin{pmatrix} -2 + 1 & 1 - 1 \\ 0 & 3 + 2 - 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}.$$

At $(3, 3)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \mathbf{J}(3, 3) = \begin{pmatrix} 3 - 6 + 1 & 3 - 1 \\ 6 - 18 & 3 + 6 - 9 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -12 & 0 \end{pmatrix}.$$

7.2. Phase-Plane Portraits Near Stationary Points. It is natural to ask when the phase-plane portrait of system (7.1) near the stationary point (x_o, y_o) looks like the phase-plane portrait of the linearized system with coefficient matrix $\mathbf{A} = \mathbf{J}(x_o, y_o)$. This turns out to be the case in some instances, but not in all.

- If the origin is *attracting* or *repelling* for the linearized system then the stationary solution (x_o, y_o) of system (7.1) will have the same property.
- If the linearized system is a *saddle*, *nodal sink*, or *nodal source*, then the phase-plane portrait of system (7.1) will look like it near the stationary point (x_o, y_o) . In particular, there will be orbits tangent to the lines passing through (x_o, y_o) in the directions of the eigenvectors of the linearized system.
- If the linearized system is a *spiral sink* or *spiral source* then the phase-plane portrait of system (7.1) will look like it near the stationary point (x_o, y_o) . In particular, the orbits will spiral in the same *clockwise* or *counterclockwise* sense.
- If system (7.1) is *conservative* and the linearized system is a *center* then the phase-plane portrait of system (7.1) will look like a center near the stationary point (x_o, y_o) . In particular, the orbits will go around it with the same *clockwise* or *counterclockwise* sense and it will be *stable*.

While this technique is easy to apply, it only gives information about the phase-plane portrait near stationary points. In order to build a better understanding of the complete phase-plane portrait, it often needs to be augmented by other techniques.

Example. Determine the stability and classify each stationary point of the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3.$$

Sketch the phase-plane portrait of the system near each stationary point.

Solution. Because $\partial_x f(x, y) + \partial_y g(x, y) = 0$, the system is Hamiltonian, and thereby is conservative. We have seen that its stationary points are

$$(-1, 0), \quad (0, 0), \quad (1, 0).$$

At $(0, 0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $p(z) = z^2 - 1 = (z + 1)(z - 1)$. Its eigenvalues thereby are -1 and 1 . Hence, the stationary point $(0, 0)$ is a saddle and is unstable. Because

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{A} - \mathbf{I} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

we see that \mathbf{A} has the eigenpairs

$$\left(-1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \quad \left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

There is one orbit that approaches $(0, 0)$ tangent to each half of the line $y = -x$. There is also one orbit that emerges from $(0, 0)$ tangent to each half of the line $y = x$.

At $(-1, 0)$ and $(1, 0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $p(z) = z^2 + 2$. Its eigenvalues thereby are the conjugate pair $\pm i\sqrt{2}$ while $a_{21} = -2 < 0$. Because the system is conservative, we conclude that the stationary points $(-1, 0)$ and $(1, 0)$ are clockwise centers and are stable.

Example. Determine the stability and classify each stationary point of the system

$$\frac{dx}{dt} = (y - x)(x - 1), \quad \frac{dy}{dt} = (3 + 2x - x^2)y.$$

Sketch the phase-plane portrait of the system near each stationary point.

Solution. We have seen that the stationary points of this system are

$$(-1, -1), \quad (0, 0), \quad (1, 0), \quad (3, 3).$$

At $(-1, -1)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -4 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $p(z) = z^2 - 2z - 8 = (z + 2)(z - 4)$. Its eigenvalues thereby are -2 and 4 . Hence, the stationary point $(-1, -1)$ is a saddle and is unstable. Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 4 & -2 \\ -4 & 2 \end{pmatrix}, \quad \mathbf{A} - 4\mathbf{I} = \begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix},$$

we see that \mathbf{A} has the eigenpairs

$$\left(-2, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right).$$

There is one orbit that approaches $(-1, -1)$ tangent to each half of the line $y = 2x + 1$. There is also one orbit that emerges from $(-1, -1)$ tangent to each half of the line $y = -x - 2$.

At $(0, 0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}.$$

Because \mathbf{A} is upper triangular, we can read off that its eigenvalues are 1 and 3 . Hence, the stationary point $(0, 0)$ is a nodal source and is repelling. Because

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix},$$

we see that \mathbf{A} has the eigenpairs

$$\left(1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right).$$

There is one orbit that emerges from $(0, 0)$ tangent to each half of the line $y = -2x$. All other orbits emerge from $(0, 0)$ tangent to one half of the x -axis.

At $(1, 0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Because \mathbf{A} is diagonal, we can read off that its eigenpairs are

$$\left(-1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Hence, the stationary point $(1, 0)$ is a saddle and is unstable. There is one orbit that approaches $(1, 0)$ tangent to each half of the x -axis. There is also one orbit that emerges from $(1, 0)$ tangent to each half of the line $x = 1$.

At $(3, 3)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \begin{pmatrix} -2 & 2 \\ -12 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $p(z) = z^2 + 2z + 24 = (z + 1)^2 + 23$. Its eigenvalues thereby are the conjugate pair $-1 \pm i\sqrt{23}$ while $a_{21} = -12 < 0$. Hence, the stationary point $(3, 3)$ is a clockwise spiral sink and is attracting.

Example. Determine the stability and classify each stationary point of the system

$$\frac{du}{dt} = -ku\sqrt{u^2 + v^2}, \quad \frac{dv}{dt} = -kv\sqrt{u^2 + v^2} - a.$$

Sketch the phase-plane portrait of the system near each stationary point.

Solution. We have seen that the only stationary point of this system is

$$\left(0, -\sqrt{a/k}\right).$$

The Jacobian matrix for this system is

$$\mathbf{J}(u, v) = \begin{pmatrix} -k\sqrt{u^2 + v^2} - k\frac{u^2}{\sqrt{u^2 + v^2}} & -k\frac{uv}{\sqrt{u^2 + v^2}} \\ -k\frac{vu}{\sqrt{u^2 + v^2}} & -k\sqrt{u^2 + v^2} - k\frac{v^2}{\sqrt{u^2 + v^2}} \end{pmatrix}.$$

The coefficient matrix of the linearized system at $(0, -\sqrt{a/k})$ is

$$\mathbf{A} = \mathbf{J}\left(0, -\sqrt{a/k}\right) = \begin{pmatrix} -\sqrt{ka} & 0 \\ 0 & -2\sqrt{ka} \end{pmatrix}.$$

Because \mathbf{A} is diagonal, we can read off that its eigenpairs are

$$\left(-\sqrt{ka}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \left(-2\sqrt{ka}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Hence, the stationary point is a nodal sink and is attracting. There is one orbit that approaches $(0, -\sqrt{a/k})$ along each half of the v -axis. All other orbits approach $(0, -\sqrt{a/k})$ tangent to one half of the line $v = -\sqrt{a/k}$.

8. APPLICATION: POPULATION DYNAMICS

Earlier in the course we studied models of population dynamics that were single first-order equations in the form

$$\frac{dp}{dt} = R(p)p,$$

where $p(t)$ is the size of the population as a function of time, and $R(p)$ models the growth rate of the population as a function of p . For a population in a closed ecosystem the growth rate is simply the birth rate minus the mortality rate. In more complicated situations it must also account for migration in to and out of the ecosystem.

The simplest such model takes $R(p) = r$ for some constant r , which has the solution $p(t) = p_I e^{rt}$ when $p(0) = p_I$. This is the so-called exponential model because when $r > 0$ its solution grows exponentially, while when $r < 0$ its solution decays exponentially. The ability of a individual to survive and reproduce depends upon the environment experienced by that individual. The assumption that $R(p)$ is constant assumes that this environment is unaffected by the number of individuals present. However once the population grows large enough there might be competition between its individuals for resources, which should reduce the growth rate. The simplest model that captures this effect takes $R(p)$ to be a linear function of p ,

$$R(p) = r - ap, \quad \text{for some constants } a \text{ and } r.$$

While this *logistic model* could be solved analytically, we found it helpful to study it with phase-line portraits. In this section we extend these ideas to study models of two interacting populations with phase-plane portraits.

8.1. Models of Two Interacting Populations. We now consider two populations that interact in a shared environment. We let the size of these populations be $p_1(t)$ and $p_2(t)$ and consider first-order autonomous models of the form

$$(8.1) \quad \frac{dp_1}{dt} = R_1(p_1, p_2)p_1, \quad \frac{dp_2}{dt} = R_2(p_1, p_2)p_2,$$

where $R_1(p_1, p_2)$ and $R_2(p_1, p_2)$ model the growth rates of the first and second populations as functions of p_1 and p_2 . We will specialize further to the case where $R_1(p_1, p_2)$ and $R_2(p_1, p_2)$ depend linearly upon p_1 and p_2 as

$$(8.2) \quad \begin{aligned} R_1(p_1, p_2) &= r_1 - a_{11}p_1 - a_{12}p_2, & \text{for some constants } a_{11}, a_{12}, \text{ and } r_1, \\ R_2(p_1, p_2) &= r_2 - a_{21}p_1 - a_{22}p_2, & \text{for some constants } a_{21}, a_{22}, \text{ and } r_2. \end{aligned}$$

This model assumes that the growth rates for each population can depend upon the size of the two populations, but nothing else. It assumes moreover that this dependence is linear. Both of these assumptions are huge oversimplifications for any ecosystem. Still, there are instances when such models correctly capture phenomena that are observed in real populations.

Remark. When such models are used in practice, p_1 and p_2 often are not the number of individuals in the respective populations, but rather are the biomass of the populations. This makes more sense than simply counting individuals because the amount of resources an individual will consume is roughly proportional to its biomass.

The six constants a_{11} , a_{12} , a_{21} , a_{22} , r_1 , and r_2 are *parameters* of the model. In practice their values are chosen so that the model fits some observed behavior. The process of choosing what data to fit and how to fit it is often called *calibrating* the model. We will not address how to calibrate such models in this course. Rather, we will be given values of these parameters and asked what behaviors are predicted by a given model.

The process of testing the model by comparing such predictions with certain observed behavior is often called *validating* the model. We will not address how to validate such models in this course. Rather, we will address how to make the kind of predictions needed for the validation process.

It is important to understand the role each of the six parameters plays in the model (8.1-8.2). We start by examining them one at a time.

The parameters r_1 and r_2 can be viewed as the bare growth rates for the respective populations. Namely, they are the rates at which the populations grow when both populations are small. When $r_k > 0$ the k^{th} population will thrive on the resources available in the environment. When $r_k < 0$ the k^{th} population will decline without some additional resource, usually the other population which it preys upon.

The parameters a_{11} and a_{22} represent how the growth rates respond to increased numbers in their respective populations. When $a_{kk} > 0$ the growth rate for the k^{th} population will reduce as that population grows. This might be the case when individuals in the population compete for the same food resource. When $a_{kk} < 0$ the growth rate for the k^{th} population will increase as that population grows. This might be the case when a greater population density increases the likelihood of individuals finding mates. However, because model (8.1-8.2) exhibits unrealistic behavior in such regimes, we will generally restrict our attention to cases where $a_{kk} \geq 0$.

Model (8.1-8.2) is interactive when the coupling parameters a_{12} and a_{21} are both nonzero. The nature of the interaction can be read off from the signs of these parameters.

- When a_{12} and a_{21} have opposite signs then one of the populations benefits from and harms the other. This would be the case, for example, when one population preys on the other. Such models are often called *predator-prey models*.
- When $a_{12} > 0$ and $a_{21} > 0$ then each population is effected adversely by the other. This would be the case, for example, when both populations compete for the same food resource. Such models are often called *competing species models*.
- When $a_{12} < 0$ and $a_{21} < 0$ then each population benefits from the other. This would be the case, for example, when the populations cooperate. Such models are sometimes called *cooperating species models*.

In this section we will focus on predator-prey and competing species models.

8.2. Predator-Prey Models. In this subsection we will study predator-prey models in the form

$$(8.3) \quad x' = (r - ax - by)x, \quad y' = (-s + cx - dy)y,$$

where the parameters b , c , r , and s are positive while a and d are nonnegative. It should be clear that x and y represent the populations of prey and predators respectively.

Example. Sketch a phase portrait for the predator-prey model

$$x' = (6 - 3y)x, \quad y' = (-15 + 5x)y.$$

Solution. Stationary points satisfy

$$0 = (6 - 3y)x, \quad 0 = (-15 + 5x)y.$$

The first equation is satisfied when either $x = 0$ or $y = 2$. When $x = 0$ the second equation can only be satisfied when $y = 0$. When $y = 2$ the second equation can only be satisfied when $x = 3$. Therefore the stationary points are

$$(0, 0), \quad (3, 2).$$

We should plot these points in the xy -plane.

Like all population models in this section, this model has semistationary solutions that lie on the x and y axes. Notice that if $y = 0$ initially then it will remain zero. Then the x' -equation reduces to $x' = 6x$, which has solution $x(t) = x_I e^{6t}$. Therefore every solution of the model with initial data on the x -axis will remain on the x -axis and have the explicit form

$$(x(t), y(t)) = (x_I e^{6t}, 0).$$

We should indicate these solutions on the x -axis with arrows pointing away from the origin. This model predicts that without predators to keep its population in check, the population of prey will grow exponentially without bound.

Similarly, notice that if $x = 0$ initially then it will remain zero. Then the y' -equation reduces to $y' = -15y$, which has solution $y(t) = y_I e^{-15t}$. Therefore every solution of the model with initial data on the y -axis will remain on the y -axis and have the explicit form

$$(x(t), y(t)) = (0, y_I e^{-15t}).$$

We should indicate these solutions on the y -axis with arrows pointing towards the origin. This model predicts that without prey to sustain it, the population of predators will die off quickly.

The orbit equation is

$$\frac{dy}{dx} = \frac{(-15 + 5x)y}{(6 - 3y)x},$$

which is separable. Its separated form is

$$\frac{5x - 15}{x} dx + \frac{3y - 6}{y} dy = 0,$$

which integrates to

$$5x - 15 \log(|x|) + 3y - 6 \log(|y|) = c.$$

This cannot be solved either for y as an explicit function of x or for x as an explicit function of y . Rather, the orbits are given implicitly by $H(x, y) = c$ where the integral $H(x, y)$ is

$$H(x, y) = 5x - 15 \log(|x|) + 3y - 6 \log(|y|).$$

This function is undefined on the x -axis and y -axis.

Because this is a population model, we only care about the first quadrant of the xy -plane. The first partial derivatives of $H(x, y)$ are

$$\partial_x H(x, y) = 5 - \frac{15}{x} = \frac{5(x-3)}{x}, \quad \partial_y H(x, y) = 3 - \frac{6}{y} = \frac{3(y-2)}{y}.$$

Therefore the stationary point $(3, 2)$ is also a critical point of the function $H(x, y)$. The Hessian matrix of $H(x, y)$ is

$$\mathbf{H}(x, y) = \begin{pmatrix} \partial_{xx}H(x, y) & \partial_{xy}H(x, y) \\ \partial_{yx}H(x, y) & \partial_{yy}H(x, y) \end{pmatrix} = \begin{pmatrix} \frac{15}{x^2} & 0 \\ 0 & \frac{6}{y^2} \end{pmatrix}.$$

Because

$$\det(\mathbf{H}(x, y)) = \frac{15}{x^2} \cdot \frac{6}{y^2}, \quad \text{tr}(\mathbf{H}(x, y)) = \frac{15}{x^2} + \frac{6}{y^2},$$

we see that these are both positive off the x -axis and y -axis. The function $H(x, y)$ thereby is strictly convex (concave up) in each quadrant. Therefore its critical point $(3, 2)$ is a global minimum of $H(x, y)$ in the first quadrant. Because the orbits of our model are the level sets of $H(x, y)$, we see that the stationary point $(3, 2)$ is a *counterclockwise center* and that counterclockwise periodic orbits fill out the first quadrant.

Example. Sketch a phase portrait for the predator-prey model

$$x' = (12 - 2x - 3y)x, \quad y' = (-15 + 5x)y.$$

Solution. Stationary points satisfy

$$0 = (12 - 2x - 3y)x, \quad 0 = (-15 + 5x)y.$$

The second equation is satisfied when either $x = 3$ or $y = 0$. When $x = 3$ the first equation can only be satisfied when $y = 2$. When $y = 0$ the first equation can be satisfied when either $x = 0$ or $x = 6$. Therefore the stationary points are

$$(0, 0), \quad (6, 0), \quad (3, 2).$$

The matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f(x, y) & \partial_y f(x, y) \\ \partial_x g(x, y) & \partial_y g(x, y) \end{pmatrix} = \begin{pmatrix} 12 - 4x - 3y & -3x \\ 5y & -15 + 5x \end{pmatrix}.$$

By evaluating this at each of the stationary points we find that the coefficient matrices for the linearizations about them are

$$\begin{aligned} @ (0, 0) \quad \mathbf{A} &= \begin{pmatrix} 12 & 0 \\ 0 & -15 \end{pmatrix}, \\ @ (6, 0) \quad \mathbf{A} &= \begin{pmatrix} -12 & -18 \\ 0 & 15 \end{pmatrix}, \\ @ (3, 2) \quad \mathbf{A} &= \begin{pmatrix} -6 & -9 \\ 10 & 0 \end{pmatrix}. \end{aligned}$$

At $(0, 0)$ the eigenvalues are 12 and -15 , so the type is a saddle. Because

$$\mathbf{A} - 12\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & -27 \end{pmatrix}, \quad \mathbf{A} + 15\mathbf{I} = \begin{pmatrix} 27 & 0 \\ 0 & 0 \end{pmatrix},$$

the eigenpairs are

$$\left(12, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \left(-15, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

At $(6, 0)$ the eigenvalues of \mathbf{A} are -12 and 15 , so the type is a saddle. Because

$$\mathbf{A} + 12\mathbf{I} = \begin{pmatrix} 0 & -18 \\ 0 & 27 \end{pmatrix}, \quad \mathbf{A} - 15\mathbf{I} = \begin{pmatrix} -27 & -18 \\ 0 & 0 \end{pmatrix},$$

the eigenpairs are

$$\left(-12, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \left(15, \begin{pmatrix} -2 \\ 3 \end{pmatrix}\right).$$

At $(3, 2)$ the characteristic polynomial of \mathbf{A} is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 6z + 90 = (z + 3)^2 + 9^2.$$

Therefore the eigenvalues of \mathbf{A} are $-3 \pm i9$. Because $a_{21} = 10 > 0$, the type is a counterclockwise spiral sink. Because its phase portrait is a spiral sink near $(3, 2)$, this predator-prey model cannot be conservative.