FIRST-ORDER SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS II: Linear Homogeneous Systems with Constant Coefficients

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Because the presentation of this material in lecture will differ from that in the book, I felt that notes that closely follow the lecture presentation might be appreciated.

Contents

3. Matrix Exponentials

3.1. **Introduction.** Consider the vector-valued initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}\,, \qquad \mathbf{x}(t_I) = \mathbf{x}_I\,, \tag{3.1}$$

where $\mathbf{A}$ is a constant, $n{\times}n$, real matrix. Let $\mathbf{\Phi}(t)$ be the natural fundamental matrix associated with (3.1) for the initial time 0. It satisfies the matrix-valued initial-value problem

$$\frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{d}t} = \mathbf{A}\mathbf{\Phi}\,, \qquad \mathbf{\Phi}(0) = \mathbf{I}\,, \tag{3.2}$$

where $\mathbf{I}$ is the $n{\times}n$ identity matrix. We assert that $\mathbf{\Phi}(t)$ satisfies

$$\begin{aligned} &\text{(i)} \quad \mathbf{\Phi}(t+s) = \mathbf{\Phi}(t)\mathbf{\Phi}(s) && \text{for every } t \text{ and } s \text{ in } \mathbb{R}\,,\\ &\text{(ii)} \quad \mathbf{\Phi}(t)\mathbf{\Phi}(-t) = \mathbf{I} && \text{for every } t \text{ in } \mathbb{R}\,. \end{aligned}$$

Assertion (i) follows because both sides satisfy the matrix-valued initial-value problem

$$\frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{d}t} = \mathbf{A}\mathbf{\Psi}\,, \qquad \mathbf{\Psi}(0) = \mathbf{\Phi}(s)\,,$$

and therefore are equal. Assertion (ii) follows by setting $s = -t$ in assertion (i) and using the fact $\mathbf{\Phi}(0) = \mathbf{I}$.

Properties (i) and (ii) look like the properties satisfied by the real-valued exponential function e^{at}, but they are satisfied by the matrix-valued function $\mathbf{\Phi}(t)$. They motivate the following definition.

Definition 2.1. The *matrix exponential* of $\mathbf{A}$ is the solution of the matrix-valued initial-value problem (3.2). It is commonly denoted either as $e^{t\mathbf{A}}$ or as $\exp(t\mathbf{A})$.

Given $e^{t\mathbf{A}}$, the solution of the initial-value problem (3.1) is given by

$$\mathbf{x}(t) = e^{(t-t_I)\mathbf{A}}\mathbf{x}_I\,,$$

while a general solution of the first-order differential system in (3.1) is given by

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c}\,, \qquad \text{where } \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

In this section and the next we will present methods by which we can compute $e^{t\mathbf{A}}$.

Remark. It is important to realize that for every $n \geq 2$ the exponential matrix $e^{t\mathbf{A}}$ is *not* the matrix obtained by exponentiating each entry! This means that

$$\text{if} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{then} \quad e^{t\mathbf{A}} \neq \begin{pmatrix} e^{ta_{11}} & e^{ta_{12}} \\ e^{ta_{21}} & e^{ta_{22}} \end{pmatrix}.$$

In particular, the initial condition of (3.2) implies that $e^{0\mathbf{A}} = \mathbf{I}$, while the matrix on the right-hand side above has every entry equal to 1 when $t = 0$.

We can see from (3.2) that for every positive integer k one has the identity

$$\mathrm{D}^k e^{t\mathbf{A}} = \mathbf{A}^k e^{t\mathbf{A}}\,, \qquad \text{where } \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,. \tag{3.3}$$

The Taylor expansion of $e^{t\mathbf{A}}$ about $t = 0$ thereby is

$$e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\, t^k \mathbf{A}^k = \mathbf{I} + t\mathbf{A} + \tfrac{1}{2}t^2\mathbf{A}^2 + \tfrac{1}{6}t^3\mathbf{A}^3 + \tfrac{1}{24}t^4\mathbf{A}^4 + \cdots\,, \tag{3.4}$$

where we define $\mathbf{A}^0 = \mathbf{I}$. Recall that the Taylor expansion of e^{at} is

$$e^{at} = \sum_{k=0}^{\infty} \frac{1}{k!}\, a^k t^k = 1 + at + \tfrac{1}{2}a^2t^2 + \tfrac{1}{6}a^3t^3 + \tfrac{1}{24}a^4t^4 + \cdots\,.$$

Motivated by the similarity of these expansions, the textbook defines $e^{t\mathbf{A}}$ by the infinite series (3.4). This approach dances around questions about the convergence of the infinite series — questions that are seldom favorites of students. We avoid such questions by defining $e^{t\mathbf{A}}$ to be the solution of the matrix-valued initial-value problem (3.2), which is also more central to the material of this course.

3.2. **Computing Matrix Exponentials.** Given any real $n\times n$ matrix $\mathbf{A}$, there are many ways to compute $e^{\mathbf{A}t}$ that are easier than evaluating the infinite series (3.4). The textbook gives a method that is based on computing the eigenvectors and (sometimes) the generalized eigenvectors of $\mathbf{A}$. This method requires different approaches depending on whether the eigenvalues of the real matrix $\mathbf{A}$ are real, complex conjugate pairs, or have multiplicity greater than one. These approaches are covered in Sections 7.5, 7.6, and 7.8 of the textbook, but these sections do not cover all the possible cases that can arise. We will cover that method in Section 4 of these notes. Here we present another method that covers all possible cases with a single approach. Moreover, this method is generally much faster to carry out than the textbook's method when n is not too large. Before giving the method, we give its three ingredients: a matrix version of the Key Identity, the Cayley-Hamilton Theorem from linear algebra, and natural fundamental sets of solutions of higher-order differential operators.

3.2.1. *Matrix Key Identity.* Just as the scalar Key Identity allowed us to construct explicit solutions to higher-order linear differential equations with constant coefficients, a matrix Key Identity will allow us to construct explicit solutions to first-order linear differential systems with a constant coefficient matrix.

Definition 3.2. Given any polynomial $p(z) = \pi_0 z^m + \pi_1 z^{m-1} + \cdots + \pi_{m-1}z + \pi_m$ and any $n\times n$ matrix $\mathbf{A}$ we define the $n\times n$ matrix $p(\mathbf{A})$ by

$$p(\mathbf{A}) = \pi_0\mathbf{A}^m + \pi_1\mathbf{A}^{m-1} + \cdots + \pi_{m-1}\mathbf{A} + \pi_m\mathbf{I}\,.$$

The polynomial $p(z)$ is said to *annihilate* $\mathbf{A}$ if $p(\mathbf{A}) = \mathbf{0}$.

It follows from (3.3) and the definition of $p(\mathbf{A})$ that

$$p(\mathrm{D})e^{t\mathbf{A}} = p(\mathbf{A})e^{t\mathbf{A}}\,. \tag{3.5}$$

This is the matrix version of the Key Identity.

Let $p(z)$ be a polynomial of degree m that annihilates $\mathbf{A}$. We see from the matrix Key Identity (3.5) that

$$p(\mathrm{D})e^{t\mathbf{A}} = p(\mathbf{A})e^{t\mathbf{A}} = \mathbf{0}\,.$$

This means that each entry of $e^{t\mathbf{A}}$ is a solution of the m^{th}-order scalar homogeneous linear differential equation with constant coefficients

$$p(\mathrm{D})e^{t\mathbf{A}} = \mathbf{0}\,. \tag{3.6}$$

Moreover, we can see from the derivative identity (3.3) that each entry of $e^{t\mathbf{A}}$ satisfies initial conditions that can be read off from

$$\mathrm{D}^k e^{t\mathbf{A}}\Big|_{t=0} = \mathbf{A}^k e^{t\mathbf{A}}\Big|_{t=0} = \mathbf{A}^k\,, \qquad \text{for } k = 0,\, 1,\, \cdots,\, m-1\,. \tag{3.7}$$

This shows that the entries of $e^{t\mathbf{A}}$ can be obtained by solving the initial-value problems (3.6–3.7) provided a polynomial can be found that annihilates $\mathbf{A}$.

3.2.2. *Cayley-Hamiltion Theorem.* We can always find a polynomial that annihilates the matrix $\mathbf{A}$ because the Cayley-Hamiltion Theorem states that one such polynomial is the *characteristic polynomial* of $\mathbf{A}$.

Definition 3.3. The *characteristic polynomial* of an $n{\times}n$ matrix $\mathbf{A}$ is

$$p_{\mathbf{A}}(z) = \det(\mathbf{I}z - \mathbf{A})\,. \tag{3.8}$$

This polyinomial has degree n and is easy to compute when n is not large. Because $\det(z\mathbf{I} - \mathbf{A}) = (-1)^n \det(\mathbf{A} - z\mathbf{I})$, this definition of $p_{\mathbf{A}}(z)$ coincides with the textbook's definition when n is even, and is its negative when n is odd. Both conventions are common. We have chosen the convention that makes $p_{\mathbf{A}}(z)$ monic. What matters most about $p_{\mathbf{A}}(z)$ is its roots and their multiplicity, which are the same for both conventions. As we will see in Section 4 of these notes, these roots are the *eigenvalues* of $\mathbf{A}$.

An important fact about characteristic polynomials is the following.

Theorem 3.1. (Cayley-Hamiltion Theorem) For every $n{\times}n$ matrix $\mathbf{A}$

$$p_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}\,. \tag{3.9}$$

In other words, every $n{\times}n$ matrix is annihilated by its characteristic polynomial.

Reason. We will not prove this theorem for general $n{\times}n$ matrices. However, it is easy to verify for $2{\times}2$ matrices by a direct calculation. Consider the general $2{\times}2$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\,.$$

Its characteristic polynomial is

$$\begin{aligned} p_{\mathbf{A}}(z) = \det(\mathbf{I}z - \mathbf{A}) &= \det\begin{pmatrix} z - a_{11} & -a_{12} \\ -a_{21} & z - a_{22} \end{pmatrix} \\ &= (z - a_{11})(z - a_{22}) - a_{21}a_{12} \\ &= z^2 - (a_{11} + a_{22})z + (a_{11}a_{22} - a_{21}a_{12}) \\ &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A})\,, \end{aligned}$$

where $\mathrm{tr}(\mathbf{A}) = a_{11} + a_{22}$ is the trace of $\mathbf{A}$. Then a direct calculation shows that

$$\begin{aligned}
p_{\mathbf{A}}(\mathbf{A}) &= \mathbf{A}^2 - (a_{11}+a_{22})\mathbf{A} + (a_{11}a_{22} - a_{21}a_{12})\mathbf{I} \\
&= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^2 - (a_{11}+a_{22}) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + (a_{11}a_{22} - a_{21}a_{12}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & (a_{11}+a_{22})a_{12} \\ (a_{11}+a_{22})a_{21} & a_{21}a_{12} + a_{22}^2 \end{pmatrix} - \begin{pmatrix} (a_{11}+a_{22})a_{11} & (a_{11}+a_{22})a_{12} \\ (a_{11}+a_{22})a_{21} & (a_{11}+a_{22})a_{22} \end{pmatrix} \\
&\quad + \begin{pmatrix} a_{11}a_{22} - a_{21}a_{12} & 0 \\ 0 & a_{11}a_{22} - a_{21}a_{12} \end{pmatrix} \\
&= \mathbf{0}\,,
\end{aligned}$$

which verifies (3.9) for 2×2 matrices. □

3.2.3. *Natural Fundamental Sets of Solutions.* The solutions of the initial-value problems (3.6–3.7) are given by

$$e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A} + \cdots + N_{m-1}(t)\mathbf{A}^{m-1}\,, \tag{3.10}$$

where $N_0(t)$, $N_1(t)$, $\cdots$, $N_{m-1}(t)$ is the natural fundamental set of solutions associated with the m^{th}-order differential operator $p(\mathrm{D})$ and the initial time 0.

Earlier in the course we showed that we can compute the natural fundamental set $N_0(t)$, $N_1(t)$, $\cdots$, $N_{m-1}(t)$ by solving the general initial-value problem

$$p(\mathrm{D})y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \quad \cdots\,, \quad y^{(m-1)}(0) = y_{m-1}\,. \tag{3.11}$$

Because $p(\mathrm{D})y = 0$ is an m^{th}-order linear equation with constant coefficients, we do this by first factoring $p(z)$ and using our recipe to generate a fundamental set of solutions $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_m(t)$. We then determine c_1, c_2, $\cdots$, c_m so that the general solution,

$$y(t) = c_1Y_1(t) + c_2Y_2(t) + \cdots + c_mY_m(t)\,,$$

satifies the general initial conditions of (3.11). By grouping the terms that multiply each y_k, we can express the solution of the general initial-value problem (3.11) as

$$y(t) = N_0(t)y_0 + N_1(t)y_1 + \cdots + N_{m-1}(t)y_{m-1}\,.$$

We can then read off $N_0(t)$, $N_1(t)$, $\cdots$, $N_{m-1}(t)$ from this expression.

3.2.4. *Our Method.* Our method to compute matrix exponentials has three steps.

1. Find a polynomial $p(z)$ that annihilates $\mathbf{A}$. Let m denote its degree.
2. Compute $N_0(t)$, $N_1(t)$, $\cdots$, $N_{m-1}(t)$, the natural fundamental set of solutions associated with the m^{th}-order differential operator $p(\mathrm{D})$ and the initial time 0.
3. Compute the matrix exponential $e^{t\mathbf{A}}$ by the formula (3.10).

By the Cayley-Hamilton Theorem, we can always find a polynomial of degree n that annihilates $\mathbf{A}$ — namely, the characteristic polynomial of $\mathbf{A}$ given by (3.8). For now, this is how you will complete Step 1. Later we will see that sometimes there is a polynomial of smaller degree that also annihilates $\mathbf{A}$. However, when the characteristic polynomial has only simple roots then no polynomial of smaller degree annihilates $\mathbf{A}$.

Once we have a polynomial of degree $m \le n$ that annihilates $\mathbf{A}$, we compute the natural fundamental set of solutions $N_0(t)$, $N_1(t)$, $\cdots$, $N_{m-1}(t)$ by solving the general initial-value problem (3.11). For now, this is how you can complete Step 2. We will present an alternative way to do it in Section 3.4.

Once we have the natural fundamental set $N_0(t)$, $N_1(t)$, $\cdots$, $N_{m-1}(t)$, we just have to apply formula (3.10) to compute $e^{t\mathbf{A}}$. If $m > 2$ this requires computing the powers of $\mathbf{A}$ that appear and doing the matrix addition. If $m = 2$ this only requires doing the matrix addition because only $\mathbf{I}$ and $\mathbf{A}$ appear in formula (3.10).

We now illustrate this method with examples.

Example. Compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Solution. Because $\mathbf{A}$ is 2×2, its characteristic polynomial is

$$p(z) = \det(\mathbf{I}z - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 5 = (z-5)(z-1).$$

Its roots are 5 and 1. The associated general initial-value problem is

$$y'' - 6y' + 5y = 0, \qquad y(0) = y_0, \quad y'(0) = y_1.$$

Its general solution is $y(t) = c_1 e^{5t} + c_2 e^{t}$. Because $y'(t) = 5c_1 e^{5t} + c_2 e^{t}$, the general initial conditions imply

$$\begin{aligned} y_0 &= y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2, \\ y_1 &= y'(0) = 5c_1 e^0 + c_2 e^0 = 5c_1 + c_2. \end{aligned}$$

Upon solving this system for c_1 and c_2 we find that

$$c_1 = \frac{y_1 - y_0}{4}, \qquad c_2 = \frac{5y_0 - y_1}{4}.$$

Therefore the solution of the general initial-value problem is

$$y(t) = \frac{y_1 - y_0}{4}\, e^{5t} + \frac{5y_0 - y_1}{4}\, e^{t} = \frac{5e^t - e^{5t}}{4}\, y_0 + \frac{e^{5t} - e^t}{4}\, y_1.$$

We then read off that the natural fundamental set is

$$N_0(t) = \frac{5e^t - e^{5t}}{4}, \qquad N_1(t) = \frac{e^{5t} - e^t}{4}.$$

Formula (3.10) then yields

$$\begin{aligned} e^{t\mathbf{A}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{A} \\ &= \frac{5e^t - e^{5t}}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{5t} - e^t}{4}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5t} + e^t & e^{5t} - e^t \\ e^{5t} - e^t & e^{5t} + e^t \end{pmatrix}. \end{aligned}$$

Example. Compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix}.$$

Solution. Because $\mathbf{A}$ is 2×2, its characteristic polynomial is

$$p(z) = \det(\mathbf{I}z - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 2z + 1 = (z+1)^2.$$

It has the double real root -1. The associated general initial-value problem is

$$y'' + 2y' + y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

Its general solution is $y(t) = c_1 e^{-t} + c_2 t e^{-t}$. Because $y'(t) = -c_1 e^{-t} - c_2 t e^{-t} + c_2 e^{-t}$, the general initial conditions imply

$$\begin{aligned} y_0 &= y(0) = c_1 e^0 + c_2 0 e^0 = c_1\,, \\ y_1 &= y'(0) = -c_1 e^0 - c_2 0 e^0 + c_2 e^0 = -c_1 + c_2\,. \end{aligned}$$

Upon solving this system for c_1 and c_2 we find that

$$c_1 = y_0\,, \qquad c_2 = y_0 + y_1\,.$$

Therefore the solution of the general initial-value problem is

$$y(t) = y_0 e^{-t} + (y_0 + y_1) t e^{-t} = (1+t)e^{-t} y_0 + t e^{-t} y_1\,.$$

We then read off that the natural fundamental set is

$$N_0(t) = (1+t)e^{-t}\,, \qquad N_1(t) = t e^{-t}\,.$$

Formula (3.10) then yields

$$\begin{aligned} e^{t\mathbf{A}} &= (1+t)e^{-t}\mathbf{I} + t e^{-t}\mathbf{A} \\ &= (1+t)e^{-t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t e^{-t}\begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix} = e^{-t}\begin{pmatrix} 1-2t & -t \\ 4t & 1+2t \end{pmatrix}\,. \end{aligned}$$

Example. Compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} 6 & -5 \\ 5 & -2 \end{pmatrix}\,.$$

Solution. Because $\mathbf{A}$ is 2×2, its characteristic polynomial is

$$\begin{aligned} p(z) &= \det(\mathbf{I}z - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 4z + 13 = (z-2)^2 + 3^2\,. \end{aligned}$$

It has conjugate roots $2 \pm i3$. The associated general initial-value problem is

$$y'' - 4y' + 13y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

Its general solution is $y(t) = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t)$. Because

$$y'(t) = 2c_1 e^{2t}\cos(3t) - 3c_1 e^{2t}\sin(3t) + 2c_2 e^{2t}\sin(3t) + 3c_2 e^{2t}\cos(3t)\,,$$

the general initial conditions imply

$$\begin{aligned} y_0 &= y(0) = c_1 e^0\cos(0) + c_2 e^0 \sin(0) = c_1\,, \\ y_1 &= y'(0) = 2c_1 e^0\cos(0) - 3c_1 e^0\sin(0) + 2c_2 e^0\sin(0) + 3c_2 e^0\cos(0) = 2c_1 + 3c_2\,. \end{aligned}$$

Upon solving this system for c_1 and c_2 we find that

$$c_1 = y_0\,, \qquad c_2 = \frac{y_1 - 2y_0}{3}\,.$$

Therefore the solution of the general initial-value problem is

$$\begin{aligned} y(t) &= y_0 e^{2t}\cos(3t) + \frac{y_1 - 2y_0}{3}\, e^{2t}\sin(3t) \\ &= \big(e^{2t}\cos(3t) - \tfrac{2}{3}e^{2t}\sin(3t)\big)\, y_0 + \tfrac{1}{3}e^{2t}\sin(3t) y_1 \,. \end{aligned}$$

We then read off that the natural fundamental set is

$$N_0(t) = e^{2t}\big(\cos(3t) - \tfrac{2}{3}\sin(3t)\big)\,, \qquad N_1(t) = e^{2t}\tfrac{1}{3}\sin(3t)\,.$$

Formula (3.10) then yields

$$\begin{aligned} e^{t\mathbf{A}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{A} \\ &= e^{2t}\big(\cos(3t) - \tfrac{2}{3}\sin(3t)\big)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{2t}\tfrac{1}{3}\sin(3t)\begin{pmatrix} 6 & -5 \\ 5 & -2 \end{pmatrix} \\ &= e^{2t}\begin{pmatrix} \cos(3t) + \tfrac{4}{3}\sin(3t) & -\tfrac{5}{3}\sin(3t) \\ \tfrac{5}{3}\sin(3t) & \cos(3t) - \tfrac{4}{3}\sin(3t) \end{pmatrix} . \end{aligned}$$

Example. Compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix} .$$

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} p(z) = \det(\mathbf{I}z - \mathbf{A}) &= \det\begin{pmatrix} z & -2 & 1 \\ 2 & z & -2 \\ -1 & 2 & z \end{pmatrix} = z^3 + 4 - 4 + 4z + 4z + z \\ &= z^3 + 9z = z(z^2+9)\,. \end{aligned}$$

Its roots are 0, $\pm i3$. The associated general initial-value problem is

$$y''' + 9y' = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \quad y''(0) = y_2\,.$$

Its general solution is $y(t) = c_1 + c_2\cos(3t) + c_3\sin(3t)$. Because

$$y'(t) = -3c_2\sin(3t) + 3c_3\cos(3t)\,, \qquad y''(t) = -9c_2\cos(3t) - 9c_3\sin(3t)\,,$$

the general initial conditions imply

$$\begin{aligned} y_0 &= y(0) = c_1 + c_2\cos(0) + c_3\sin(0) = c_1 + c_2\,, \\ y_1 &= y'(0) = -3c_2\sin(0) + 3c_3\cos(0) = 3c_3\,, \\ y_2 &= y''(0) = -9c_2\cos(0) - 9c_3\sin(0) = -9c_2\,, \end{aligned}$$

Upon solving this system for c_1, c_2, and c_3 we find that

$$c_1 = \frac{9y_0 + y_2}{9}\,, \qquad c_2 = -\frac{y_2}{9}\,, \qquad c_3 = \frac{y_1}{3}\,.$$

Therefore the solution of the general initial-value problem is

$$\begin{aligned} y(t) &= \frac{9y_0 + y_2}{9} - \frac{y_2}{9}\cos(3t) + \frac{y_1}{3}\sin(3t) \\ &= y_0 + \frac{\sin(3t)}{3}\, y_1 + \frac{1-\cos(3t)}{9}\, y_2 \,. \end{aligned}$$

We then read off that the natural fundamental set is

$$N_0(t) = 1\,, \qquad N_1(t) = \frac{\sin(3t)}{3}\,, \qquad N_2(t) = \frac{1-\cos(3t)}{9}\,.$$

Formula (3.10) then yields

$$\begin{aligned} e^{t\mathbf{A}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{A} + N_2(t)\mathbf{A}^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sin(3t)}{3}\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix} + \frac{1-\cos(3t)}{9}\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sin(3t)}{3}\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix} + \frac{1-\cos(3t)}{9}\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{4+5\cos(3t)}{9} & \frac{2-2\cos(3t)+6\sin(3t)}{9} & \frac{4-4\cos(3t)-3\sin(3t)}{9} \\ \frac{2-2\cos(3t)-6\sin(3t)}{9} & \frac{1+8\cos(3t)}{9} & \frac{2-2\cos(3t)+6\sin(3t)}{9} \\ \frac{4-4\cos(3t)+3\sin(3t)}{9} & \frac{2-2\cos(3t)+6\sin(3t)}{9} & \frac{4+5\cos(3t)}{9} \end{pmatrix} . \end{aligned}$$

Remark. The above examples show that, once the natural fundamental set of solutions is found for the associated higher-order equation, employing formula (3.10) is straightforward. It requires only computing $\mathbf{A}^k$ up to $k = m-1$ and some addition. For $m \geq 2$ this requires $(m-2)n^3$ multiplications, which grows fast as m and n get large. (Often $m = n$.) However, for small systems like the ones you will face in this course, it is generally the fastest method.

3.3. **Two-by-Two Matrix Exponentials.** We will now apply our method to derive simple formulas for the exponential of any real matrix $\mathbf{A}$ that is annihilated by a quadratic polynomial $p(z) = z^2 + \pi_1 z + \pi_2$. This includes the general 2×2 real matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ,$$

which is annihilated by its characteristic polynomial,

$$p(z) = \det(\mathbf{I}z - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A})\,.$$

Upon completing the square we see that

$$p(z) = z^2 + \pi_1 z + \pi_2 = (z-\mu)^2 - \delta\,,$$

where the mean μ and discriminant δ are given by

$$\mu = -\frac{\pi_1}{2}\,, \qquad \delta = \mu^2 - \pi_2\,.$$

There are three cases which are distinguished by the sign of δ.

- If $\delta > 0$ then $p(z)$ has the simple real roots $\mu - \nu$ and $\mu + \nu$ where $\nu = \sqrt{\delta}$. In this case the natural fundamental set of solutions is found to be

$$N_0(t) = e^{\mu t}\cosh(\nu t) - \mu e^{\mu t}\frac{\sinh(\nu t)}{\nu}\,, \qquad N_1(t) = e^{\mu t}\frac{\sinh(\nu t)}{\nu}\,. \tag{3.12}$$

- If $\delta < 0$ then $p(z)$ has the complex conjugate roots $\mu - i\nu$ and $\mu + i\nu$ where $\nu = \sqrt{-\delta}$. In this case the natural fundamental set of solutions is found to be

$$N_0(t) = e^{\mu t}\cos(\nu t) - \mu e^{\mu t}\frac{\sin(\nu t)}{\nu}\,, \qquad N_1(t) = e^{\mu t}\frac{\sin(\nu t)}{\nu}\,. \tag{3.13}$$

- If $\delta = 0$ then $p(z)$ has the double real root μ. In this case the natural fundamental set of solutions is found to be

$$N_0(t) = e^{\mu t} - \mu e^{\mu t} t\,, \qquad N_1(t) = e^{\mu t} t\,. \tag{3.14}$$

Notice that (3.14) is the limiting case of both (3.12) and (3.13) as $\nu \to 0$ because

$$\lim_{\nu\to 0}\cosh(\nu t) = \lim_{\nu\to 0}\cos(\nu t) = 1\,, \qquad \lim_{\nu\to 0}\frac{\sinh(\nu t)}{\nu} = \lim_{\nu\to 0}\frac{\sin(\nu t)}{\nu} = t\,.$$

Because $p(z)$ is quadratic, $m = 2$ in formula (3.10) and it reduces to

$$e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A}\,.$$

When the natural fundamental sets (3.12), (3.13), and (3.14) respectively are plugged into this formula, we obtain the following formulas.

- If $\delta > 0$ then $p(z)$ has the simple real roots $\mu - \nu$ and $\mu + \nu$ where $\nu = \sqrt{\delta}$. In this case

$$e^{t\mathbf{A}} = e^{\mu t}\left[\cosh(\nu t)\mathbf{I} + \frac{\sinh(\nu t)}{\nu}(\mathbf{A} - \mu\mathbf{I})\right]. \tag{3.15}$$

- If $\delta < 0$ then $p(z)$ has the complex conjugate roots $\mu - i\nu$ and $\mu + i\nu$ where $\nu = \sqrt{-\delta}$. In this case

$$e^{t\mathbf{A}} = e^{\mu t}\left[\cos(\nu t)\mathbf{I} + \frac{\sin(\nu t)}{\nu}(\mathbf{A} - \mu\mathbf{I})\right]. \tag{3.16}$$

- If $\delta = 0$ then $p(z)$ has the double real root μ. In this case

$$e^{t\mathbf{A}} = e^{\mu t}\big[\mathbf{I} + t\,(\mathbf{A} - \mu\mathbf{I})\big]\,. \tag{3.17}$$

For any matrix that is annihialted by a quadratic polynomial you will find that it is faster to apply these formulas than to set-up and solve the general initial-value problem for $N_0(t)$ and $N_1(t)$. These formulas are eazy to remember. Notice that formulas (3.15) and (3.16) are similar — the first uses hyperbolic functions when $p(z)$ has simple real roots, while the second uses trigonometric functions when $p(z)$ has complex conjugate roots. Formula (3.17) is the limiting case of both (3.15) and (3.16) as $\nu \to 0$.

Remark. For 2×2 matrices $\mu = \frac{1}{2}\,\mathrm{tr}(\mathbf{A})$ while $\mathrm{tr}(\mathbf{I}) = 2$. In that case the trace of the matrix $\mathbf{A} - \mu\mathbf{I}$ that appears in each of the formulas (3.15–3.17) is zero. You should use this fact as a check whenever you use these formulas to compute $e^{t\mathbf{A}}$ for 2×2 matrices.

We will illustrate this approach on the same 2×2 matrices used in the last subsection.

Example. Compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} .$$

Solution. Because $\mathbf{A}$ is 2×2, its characteristic polynomial is

$$p(z) = \det(\mathbf{I}z - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 5 = (z-3)^2 - 4 .$$

It has the real roots 3 ± 2. By (3.15) with $\mu = 3$ and $\nu = 2$ we see that

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t}\left[\cosh(2t)\mathbf{I} + \frac{\sinh(2t)}{2}(\mathbf{A} - 3\mathbf{I})\right] \\ &= e^{3t}\left[\cosh(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh(2t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = e^{3t}\begin{pmatrix} \cosh(2t) & \sinh(2t) \\ \sinh(2t) & \cosh(2t) \end{pmatrix} . \end{aligned}$$

Remark. At first glance this may not look like the same answer that we got in the last section — but it is! It is simply expressed in terms of hyperbolic functions rather than just exponentials. Either form of the answer is correct.

Example. Compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix} .$$

Solution. Because $\mathbf{A}$ is 2×2, its characteristic polynomial is

$$p(z) = \det(\mathbf{I}z - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 2z + 1 = (z+1)^2 .$$

It has the double real root -1. By (3.17) with $\mu = -1$ we see that

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-t}\big[\mathbf{I} + t\,(\mathbf{A} + \mathbf{I})\big] \\ &= e^{-t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\right] = e^{-t}\begin{pmatrix} 1-2t & -t \\ 4t & 1+2t \end{pmatrix} . \end{aligned}$$

Example. Compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} 6 & -5 \\ 5 & -2 \end{pmatrix} .$$

Solution. Because $\mathbf{A}$ is 2×2, its characteristic polynomial is

$$\begin{aligned} p(z) &= \det(\mathbf{I}z - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 4z + 13 = (z-2)^2 + 3^2 . \end{aligned}$$

It has the conjugate roots $2 \pm i3$. By (3.16) with $\mu = 2$ and $\nu = 3$ we see that

$$\begin{aligned} e^{t\mathbf{A}} &= e^{2t}\left[\cos(3t)\mathbf{I} + \frac{\sin(3t)}{3}(\mathbf{A} - 2\mathbf{I})\right] \\ &= e^{2t}\left[\cos(3t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(3t)}{3}\begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix}\right] \\ &= e^{2t}\begin{pmatrix} \cos(2t) + \frac{4}{3}\sin(3t) & -\frac{5}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) & \cos(3t) - \frac{4}{3}\sin(3t) \end{pmatrix} . \end{aligned}$$

3.4. **Natural Fundamental Sets from Green Functions.** If $\mathbf{A}$ is an $n \times n$ matrix that is annihilated by a polynomial of degree $m > 2$ and n is not too large then the hardest step in our method for computing $e^{t\mathbf{A}}$ is generating the natural fundamental set $N_0(t)$, $N_1(t)$, $\cdots$, $N_{m-1}(t)$ by solving the general initial-value problem (3.11). Here we present an alternative way to generate these solutions.

Observe that in each of the natural fundamental sets of solutions given by (3.12), (3.13), and (3.14), the solutions $N_0(t)$ and $N_1(t)$ are related by

$$N_0(t) = N_1'(t) - 2\mu N_1(t) .$$

This is an instance of a more general fact. For the m^{th}-order equation

$$p(\mathrm{D})y = 0 , \qquad \text{where} \quad p(z) = z^m + \pi_1 z^{m-1} + \cdots + \pi_{m-1} z + \pi_m , \tag{3.18}$$

one can generate its entire natural fundamental set of solutions from the *Green function* $g(t)$ associated with (3.18). Recall that the Green function $g(t)$ satisfies the initial-value problem

$$p(\mathrm{D})g = 0 , \qquad g(0) = g'(0) = \cdots = g^{(m-2)}(0) = 0 , \quad g^{(m-1)}(0) = 1 . \tag{3.19}$$

The natural fundamental set of solutions is then given by the recipe

$$\begin{aligned} N_{m-1}(t) &= g(t) , \\ N_{m-2}(t) &= N_{m-1}'(t) + \pi_1 g(t) , \\ &\ \vdots \\ N_0(t) &= N_1'(t) + \pi_{m-1} g(t) . \end{aligned} \tag{3.20}$$

This entire set thereby is generated by the solution g of the initial-value problem (3.19).

Remark. It is clear that solving the initial-value problem (3.19) for the Green function $g(t)$ requires less algebra than solving the general initial-value problem (3.11) directly for $N_0(t)$, $N_1(t)$, $\cdots$, $N_{m-1}(t)$. The trade-off is that we now have to compute the $m-1$ derivatives required by recipe (3.20).

Example. Compute the natural fundamental set of solutions to the equation

$$y''' + 9y' = 0 .$$

Solution. By (3.19) the Green function $g(t)$ satisfies the initial-value problem

$$g''' + 9g' = 0 , \qquad g(0) = g'(0) = 0 , \quad g''(0) = 1 .$$

The characteristic polynomial of this equation is $p(z) = z^3 + 9z$, which has roots 0, $\pm i3$. Therefore we seek a solution in the form

$$g(t) = c_1 + c_2 \cos(3t) + c_3 \sin(3t) .$$

Because

$$g'(t) = -3c_2 \sin(3t) + 3c_3 \cos(3t) , \qquad g''(t) = -9c_2 \cos(3t) - 9c_3 \sin(3t) ,$$

the initial conditions for $g(t)$ then yield the algebraic system

$$g(0) = c_1 + c_2 = 0 , \qquad g'(0) = 3c_3 = 0 , \qquad g''(0) = -9c_2 = 1 .$$

Solving this system gives $c_1 = \frac{1}{9}$, $c_2 = -\frac{1}{9}$, and $c_3 = 0$, whereby the Green function is

$$g(t) = \frac{1 - \cos(3t)}{9} \,.$$

Because $p(z) = z^3 + 9z$, we read off from (3.18) that $\pi_1 = 0$, $\pi_2 = 9$, and $\pi_3 = 0$. Then by recipe (3.20) the natural fundamental set of solutions is given by

$$\begin{aligned}
N_2(t) &= g(t) = \frac{1 - \cos(3t)}{9} \,, \\
N_1(t) &= N_2'(t) + 0 \cdot g(t) = \frac{\sin(3t)}{3} \,, \\
N_0(t) &= N_1'(t) + 9 \cdot g(t) = \cos(3t) + 9 \cdot \frac{1 - \cos(3t)}{9} = 1 \,.
\end{aligned}$$

Remark. Earlier we had computed the above fundamental set by solving the general initial-value problem (3.11). You should compare that calculation with the one above.

Example. Given the fact that $p(z) = z^3 - 4z$ annihilates $\mathbf{A}$, compute $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} .$$

Solution. Because we are told that $p(z) = z^3 - 4z$ annihilates $\mathbf{A}$, we do not have to compute the characteristic polynomial of $\mathbf{A}$. By (3.19) the Green function $g(t)$ satisfies the initial-value problem

$$g''' - 4g' = 0 \,, \qquad g(0) = g'(0) = 0 \,, \quad g''(0) = 1 \,.$$

The characteristic polynomial of this equation is $p(z) = z^3 - 4z$, which has roots 0, ± 2. Therefore we seek a solution in the form

$$g(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t} \,.$$

Because

$$g'(t) = 2c_2 e^{2t} - 2c_3 e^{-2t} \,, \qquad g''(t) = 4c_2 e^{2t} + 4c_3 e^{-2t} \,,$$

the initial conditions for $g(t)$ then yield the algebraic system

$$g(0) = c_1 + c_2 + c_3 = 0 \,, \qquad g'(0) = 2c_2 - 2c_3 = 0 \,, \qquad g''(0) = 4c_2 + 4c_3 = 1 \,.$$

The solution of this system is $c_1 = -\frac{1}{4}$ and $c_2 = c_3 = \frac{1}{8}$, whereby the Green function is

$$g(t) = -\tfrac{1}{4} + \tfrac{1}{8} e^{2t} + \tfrac{1}{8} e^{-2t} = \tfrac{1}{4}\big(\cosh(2t) - 1\big) \,.$$

Because $p(z) = z^3 - 4z$, we read off from (3.18) that $\pi_1 = 0$, $\pi_2 = -4$, and $\pi_3 = 0$. Then by recipe (3.20) the natural fundamental set of solutions is given by

$$\begin{aligned}
N_2(t) &= g(t) = \frac{\cosh(2t) - 1}{4}\,, \\
N_1(t) &= N_2'(t) + 0 \cdot g(t) = \frac{\sinh(2t)}{2}\,, \\
N_0(t) &= N_1'(t) - 4 \cdot g(t) = \cosh(2t) - 4 \cdot \frac{\cosh(2t) - 1}{4} = 1\,.
\end{aligned}$$

Given this set of solutions, formula (3.10) yields

$$\begin{aligned}
e^{t\mathbf{A}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{A} + N_2(t)\mathbf{A}^2 \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{\sinh(2t)}{2} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} + \frac{\cosh(2t) - 1}{4} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{\sinh(2t)}{2} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} + \frac{\cosh(2t) - 1}{4} \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 + \cosh(2t) & \sinh(2t) & \cosh(2t) - 1 & \sinh(2t) \\ \sinh(2t) & 1 + \cosh(2t) & \sinh(2t) & \cosh(2t) - 1 \\ \cosh(2t) - 1 & \sinh(2t) & 1 + \cosh(2t) & \sinh(2t) \\ \sinh(2t) & \cosh(2t) - 1 & \sinh(2t) & 1 + \cosh(2t) \end{pmatrix}.
\end{aligned}$$

Remark. The polynomial $p(z) = z^3 - 4z$ that was used in the previous example is not the characteristic polynomial of $\mathbf{A}$. You can check that $p_{\mathbf{A}}(z) = z^4 - 4z^2$. We saved a lot of work by using $p(z) = z^3 - 4z$ as the annihilating polynomial rather than $p_{\mathbf{A}}(z) = z^4 - 4z^2$ because $p(z)$ has a smaller degree.

Given the characteristic polynomial $p_{\mathbf{A}}(z)$ of any matrix $\mathbf{A}$, we can seek a polynomial $p(z)$ of smaller degree that also annihilates $\mathbf{A}$ guided by the following facts.

- Every polynomial $p(z)$ that annihilates $\mathbf{A}$ will have the same roots as $p_{\mathbf{A}}(z)$, but these roots might have smaller multiplicity. If $p_{\mathbf{A}}(z)$ has roots that are not simple then it pays to seek a polynomial of smaller degree that annihilates $\mathbf{A}$.
- If $p(z)$ has simple roots then it has the smallest degree possible for a polynomial that annihilates $\mathbf{A}$.
- If $\mathbf{A}$ is either symmetric ($\mathbf{A}^T = \mathbf{A}$), skew-symmetric ($\mathbf{A}^T = -\mathbf{A}$), or normal ($\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$) then there is a polynomial that annihilates $\mathbf{A}$ with simple roots.

In the last example, because $\mathbf{A}$ is symmetric and $p_{\mathbf{A}}(z)$ has roots -2, 0, 0, and 2, we know that $p(z) = (z + 2)z(z - 2) = z^3 - 4z$ is an annihilating polynomial with simple roots. It thereby has the smallest degree possible for a polynomial that annihilates $\mathbf{A}$.

Justification of Recipe (3.20). The following justification of recipe (3.20) is included for completeness. It was not covered in lecture and you do not need to know it. However, you should find the recipe itself quite useful.

Suppose that the Green function $g(t)$ satisfies the initial-value problem (3.19) while the functions $N_0(t)$, $N_1(t)$, $\cdots$, $N_{m-1}(t)$ are defined by recipe (3.20). We want to show that for each $j = 0, 1, \cdots, m-1$ the function $N_j(t)$ is the solution of

$$p(\mathrm{D})y = y^{(m)} + \pi_1 y^{(m-1)} + \pi_2 y^{(m-2)} + \cdots + \pi_{m-1} y' + \pi_m y = 0\,, \tag{3.21}$$

that satisfies the initial conditions

$$N_j^{(k)}(0) = \delta_{jk} \quad \text{for } k = 0, 1, \cdots, m-1\,. \tag{3.22}$$

Here δ_{jk} is the Kronecker delta, which is defined by

$$\delta_{jk} = \begin{cases} 1 & \text{when } j = k\,, \\ 0 & \text{when } j \neq k\,. \end{cases}$$

Because $N_{m-1}(t) = g(t)$, we see from (3.19) that it satisfies the differential equation (3.21) and the initial conditions (3.22) for $j = m-1$. The key step is to show that if $N_j(t)$ satisfies (3.21) and (3.22) for some positive $j < m$ then so does $N_{j-1}(t)$. Once this step is done then we can argue that because $N_{m-1}(t)$ has these properties, so does $N_{m-2}(t)$, which implies so does $N_{m-3}(t)$, and so on down to $N_0(t)$.

We now prove the key step. Suppose that $N_j(t)$ satisfies (3.21) and (3.22) for some positive $j < m$. Because $N_j(t)$ satisfies (3.21), so does $N_j'(t)$. Because $N_{j-1}(t)$ is given by recipe (3.20) as a linear combination of $N_j'(t)$ and $g(t)$, while $N_j'(t)$ and $g(t)$ satisfy (3.21), we see that $N_{j-1}(t)$ also satisfies the differential equation (3.21). The only thing that remains to be checked is that $N_{j-1}(t)$ satifies the initial conditions (3.22).

Because $N_j(t)$ satisfies (3.21) and (3.22) we see that

$$\begin{aligned} 0 = p(\mathrm{D}) N_j(t)\big|_{t=0} &= N_j^{(m)}(0) + \sum_{k=0}^{m-1} \pi_{m-k} N_j^{(k)}(0) \\ &= N_j^{(m)}(0) + \sum_{k=0}^{m-1} \pi_{m-k} \delta_{jk} = N_j^{(m)}(0) + \pi_{m-j}\,. \end{aligned} \tag{3.23}$$

Because $N_{j-1}(t)$ is given by recipe (3.20) to be $N_{j-1}(t) = N_j'(t) + \pi_{m-j} g(t)$, we evaluate the $(k-1)^{st}$ derivative of this relation at $t = 0$ to obtain

$$N_{j-1}^{(k-1)}(0) = N_j^{(k)}(0) + \pi_{m-j} g^{(k-1)}(0) \quad \text{for } k = 1, 2, \cdots, m\,.$$

Because $g(t)$ satisfies the initial condtions in (3.19), we see from (3.22) that this becomes

$$N_{j-1}^{(k-1)}(0) = \delta_{jk} \quad \text{for } k = 1, 2, \cdots, m-1\,,$$

while we see from (3.23) that for $k = m$ it becomes

$$N_{j-1}^{(m-1)}(0) = N_j^{(m)}(0) + \pi_{m-j} = 0\,.$$

The initial conditions (3.22) thereby hold for $N_{j-1}(t)$. This completes the proof of the key step, which completes the justification of recipe (3.20). □

3.5. **Matrix Exponential Formula.** We now present a formula for the exponential of any $n\times n$ matrix $\mathbf{A}$. This formula will be the basis for a method presented in the next subsection that computes $e^{t\mathbf{A}}$ efficiently when n is not too large. It is also related to the method of generalized eigenvectors that computes $e^{t\mathbf{A}}$ efficiently when n is large.

Let be $p(z)$ be any monic polynomial of degree $m \leq n$ that annihilates $\mathbf{A}$. The formula for $e^{t\mathbf{A}}$ will be given explicitly in terms of $\mathbf{A}$ and the roots of $p(z)$. Suppose that $p(z)$ has l roots λ_1, λ_2, $\cdots$, λ_l (possibly complex) with multiplicities m_1, m_2, $\cdots$, m_l respectively. This means that $p(z)$ has the factored form

$$p(z) = \prod_{k=1}^{l} (z - \lambda_k)^{m_k}\,, \tag{3.24}$$

and that $m_1 + m_2 + \cdots + m_l = m$. Here we understand that $\lambda_j \neq \lambda_k$ if $j \neq k$. Then

$$e^{t\mathbf{A}} = \sum_{k=1}^{l} e^{\lambda_k t} \left(\mathbf{I} + t\,(\mathbf{A} - \lambda_k \mathbf{I}) + \cdots + \frac{t^{m_k - 1}}{(m_k - 1)!}\,(\mathbf{A} - \lambda_k \mathbf{I})^{m_k - 1} \right) \mathbf{Q}_k\,, \tag{3.25a}$$

where the $n\times n$ matrices $\mathbf{Q}_k$ are defined by

$$\mathbf{Q}_k = \prod_{\substack{j=1\\ j\neq k}}^{l} \left(\frac{\mathbf{A} - \lambda_j \mathbf{I}}{\lambda_k - \lambda_j} \right)^{m_j} \qquad \text{for every } k = 1,\, 2,\, \cdots,\, l\,. \tag{3.25b}$$

Before we justify formula (3.25), let us consider its structure. It can be recast as $e^{t\mathbf{A}} = h(\mathbf{A})$ where $h(z)$ is the time-dependent polynomial

$$h(z) = \sum_{k=1}^{l} e^{\lambda_k t} \left(1 + t\,(z - \lambda_k) + \cdots + \frac{t^{m_k - 1}}{(m_k - 1)!}\,(z - \lambda_k)^{m_k - 1} \right) q_k(z)\,, \tag{3.26a}$$

with the time-independent polynomials $q_k(z)$ defined by

$$q_k(z) = \prod_{\substack{j=1\\ j\neq k}}^{l} \left(\frac{z - \lambda_j}{\lambda_k - \lambda_j} \right)^{m_j} \qquad \text{for every } k = 1,\, 2,\, \cdots,\, l\,. \tag{3.26b}$$

Because the degree of each $q_k(z)$ is $m - m_k$, it follows from (3.26a) that the degree of $h(z)$ is at most $m - 1$.

It can be checked that for every $k = 1,\, 2,\, \cdots,\, l$ the polynomial $h(z)$ satisfies the m_k relations

$$h(\lambda_k) = e^{\lambda_k t}\,, \quad h'(\lambda_k) = t\, e^{\lambda_k t}\,, \quad \cdots\,, \quad h^{(m_k - 1)}(\lambda_k) = t^{m_k - 1} e^{\lambda_k t}\,. \tag{3.27}$$

These relations state that the functions $h(z)$ and e^{zt} and their derivatives with respect to z though order $m_k - 1$ agree at $z = \lambda_k$. Because these m_k relations hold for each k, this gives a total of m relations that $h(z)$ satisfies. The theory of *Hermite interpolation* states that these m relations uniquely specify the polynomial $h(z)$ at every time t, which is the called associated *Hermite interpolant* of e^{zt}.

Justification of Formula (3.25). Let $\mathbf{\Phi}(t)$ denote the right-hand side of (3.25a). We will show that $\mathbf{\Phi}(t)$ satisfies the matrix-valued initial-value problem (3.2), and thereby conclude that $\mathbf{\Phi}(t) = e^{t\mathbf{A}}$ as asserted by formula (3.25).

We first show that $\mathbf{\Phi}(0) = \mathbf{I}$. By setting $t = 0$ in the right-hand side of (3.25a) we get $\mathbf{\Phi}(0) = h(\mathbf{A})$ where the polynomial $h(z)$ is obtained by setting $t = 0$ in formula (3.26a) — namely,

$$h(z) = \sum_{k=1}^{l} q_k(z)\,,$$

where the polynomials $q_k(z)$ are defined by (3.26b). The polynomial $h(z)$ has degree less than m. By (3.27), for every $k = 1, 2, \cdots, l$ this polynomial satisfies the m_k conditions

$$h(\lambda_k) = 1\,, \quad h'(\lambda_k) = 0\,, \quad \cdots\,, \quad h^{(m_k-1)}(\lambda_k) = 0\,.$$

Because the constant polynomial 1 satisfies these conditions and has degree less than m, we conclude by the uniqueness of the Hermite interpolant that $h(z) = 1$. Therefore $\mathbf{\Phi}(0) = h(\mathbf{A}) = \mathbf{I}$, which is the initial condition of (3.2).

Next we show that $\mathbf{\Phi}(t)$ also satisfies the differential system of (3.2). Because $p(z)$ annihilates $\mathbf{A}$ and has the factored form (3.24), it follows that

$$(\mathbf{A} - \lambda_k\mathbf{I})^{m_k}\mathbf{Q}_k = \mathbf{0} \qquad \text{for every } k = 1, 2, \cdots, l\,.$$

By using this fact we see that

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{d}t}(t) &= \sum_{k=1}^{l} \lambda_k e^{\lambda_k t}\left(\mathbf{I} + t\,(\mathbf{A} - \lambda_k\mathbf{I}) + \cdots + \frac{t^{m_k-1}}{(m_k-1)!}\,(\mathbf{A} - \lambda_k\mathbf{I})^{m_k-1}\right)\mathbf{Q}_k \\
&\quad + \sum_{k=1}^{l} e^{\lambda_k t}\left((\mathbf{A} - \lambda_k\mathbf{I}) + \cdots + \frac{t^{m_k-2}}{(m_k-2)!}\,(\mathbf{A} - \lambda_k\mathbf{I})^{m_k-1}\right)\mathbf{Q}_k \\
&= \sum_{k=1}^{l} \lambda_k e^{\lambda_k t}\left(\mathbf{I} + t\,(\mathbf{A} - \lambda_k\mathbf{I}) + \cdots + \frac{t^{m_k-1}}{(m_k-1)!}\,(\mathbf{A} - \lambda_k\mathbf{I})^{m_k-1}\right)\mathbf{Q}_k \\
&\quad + \sum_{k=1}^{l} e^{\lambda_k t}(\mathbf{A} - \lambda_k\mathbf{I})\left(\mathbf{I} + t\,(\mathbf{A} - \lambda_k\mathbf{I}) + \cdots + \frac{t^{m_k-1}}{(m_k-1)!}\,(\mathbf{A} - \lambda_k\mathbf{I})^{m_k-1}\right)\mathbf{Q}_k \\
&= \mathbf{A}\mathbf{\Phi}(t)\,.
\end{aligned}$$

Therefore $\mathbf{\Phi}(t)$ also satisfies the differential system of (3.2). Because $\mathbf{\Phi}(t)$ satisfies the matrix-valued initial-value problem (3.2), we conclude that $\mathbf{\Phi}(t) = e^{t\mathbf{A}}$, thereby proving formula (3.25).

Remark. Naively plugging $\mathbf{A}$ into formula (3.25) is an extremely inefficient way to evaluate $e^{t\mathbf{A}}$. This is because for each $k = 1, \cdots, l$ it requires $m_k - 1$ matrix multiplications to compute $(\mathbf{A} - \lambda_k\mathbf{I})^j$ for $j = 2, \cdots, m_k$, which adds to $m - l$ matrix multiplications, while formula (3.25b) requires $l - 2$ matrix multiplications to compute each $\mathbf{Q}_k$, which adds to $l^2 - 2l$. Naively evaluating formula (3.25) thereby requires at least $l^2 - 3l + m$ matrix multiplications. Because each matrix multiplication requires n^3 multiplications of numbers, naively evaluating formula (3.25) when $l = m = n$ therefore requires $n^5 - 2n^4$ multiplications of numbers. This is 81 multiplications for $n = 3$ and 512 multiplications for $n = 4$ — ouch!

3.6. **Hermite Interpolation Methods.** We can compute $e^{t\mathbf{A}}$ from formula (3.25) more efficiently by adapting methods for evaluating the Hermite interpolant $h(z)$ efficiently. Moreover, these methods can be applied to compute other functions of matrices like high powers.

3.6.1. *Computing Hermite Interpolants.* Given any function $f(z)$ of the complex variable z. An Hermite interpolant $h(z)$ of $f(z)$ is a polynomial of degree at most $m-1$ that satisfies m conditions. These m conditions are encoded by a list of m points z_1, z_2, $\cdots$, z_m. These are listed with multiplicity and clustered so that

$$\text{if } z_j = z_k \text{ for some } j < k \text{ then } z_j = z_{j+1} = \cdots = z_{k-1} = z_k\,.$$

The interpolation conditions take the form for every $1 \leq j \leq k \leq m$

$$h^{(k-j)}(z_k) = f^{(k-j)}(z_k) \quad \text{if } z_j = z_k\,. \tag{3.28}$$

These relations state that $h(z)$ and $f(z)$ agree at $z = z_k$.

The way to evaluate $h(z)$ efficiently is to first express it in the Newton form

$$h(z) = \sum_{k=0}^{m-1} h_{k+1}\, p_k(z)\,, \tag{3.29a}$$

where the polynomials $p_k(z)$ are defined by

$$p_0(z) = 1\,, \qquad p_k(z) = \prod_{j=1}^{k}(z - z_j) \quad \text{for } k = 1, \cdots, m-1\,. \tag{3.29b}$$

Notice that each $p_k(z)$ is a monic polynomial of degree k. The coefficients h_k are determined by completing a so-called divided-difference table of the form

$$\begin{array}{cccccccccc}
z_1 & h[z_1] & & & & & & & & \\
 & & \searrow & & & & & & & \\
z_2 & h[z_2] & \to & h[z_1,z_2] & & & & & & \\
 & & \searrow & & \searrow & & & & & \\
z_3 & h[z_3] & \to & h[z_2,z_3] & \to & h[z_1,z_2,z_3] & & & & \\
 & & \searrow & & \searrow & & \searrow & & & \\
\vdots & \vdots & & \vdots & & \vdots & & \ddots & & \\
 & & \searrow & & \searrow & & \searrow & & \searrow & \\
z_m & h[z_m] & \to & h[z_{m-1},z_m] & \to & h[z_{m-2},z_{m-1},z_m] & \to & \cdots & \to & h[z_1,\cdots,z_m]\,,
\end{array}$$

where the first column of the table is seeded with the entries $h[z_k] = f(z_k)$ and entries of subsequent columns are obtain for every $1 \leq j < k \leq m$ from the divided-difference formula

$$h[z_j,\cdots,z_k] = \begin{cases} \dfrac{h[z_{j+1},\cdots,z_k] - h[z_j,\cdots,z_{k-1}]}{z_k - z_j}\,, & \text{if } z_j \neq z_k\,; \\[2ex] \dfrac{1}{(k-j)!} f^{(k-j)}(z_k)\,, & \text{if } z_j = z_k\,. \end{cases} \tag{3.29c}$$

Notice that entry $h[z_j,\cdots,z_k]$ depends only on the entry to its left, and the entry above that one. This dependence is indicated by the arrows in the table. After the table is completed, the coefficients h_k in (3.29a) are read off from the top entries of each column as

$$h_k = h[z_1, \cdots, z_k] \qquad \text{for } k = 1, \cdots, m\,. \tag{3.29d}$$

3.6.2. *Application to Matrix Exponentials.* We can now evaluate formula (3.25) by applying recipe (3.29) to the function $f(z) = e^{tz}$. Let $\mathbf{A}$ be an $n \times n$ real matrix and $p(z)$ be a polyinomial of degree $m \leq n$ that annihilates $\mathbf{A}$. Let $\{z_1, \cdots, z_m\}$ be the roots of $p(z)$ listed with multiplicity and clustered so that

$$\text{if } z_j = z_k \text{ for some } j < k \text{ then } z_j = z_{j+1} = \cdots = z_{k-1} = z_k\,.$$

Compute the divided-difference table whose first column is seeded with $h[z_k] = e^{tz_k}$ and subsequent columns are given by (3.29c). Read off the functions $h_1(t)$, $h_2(t)$, $\cdots$, $h_m(t)$ from the top entries of each column as prescribed by (3.29d). Then set

$$e^{t\mathbf{A}} = h(\mathbf{A}) = \sum_{k=0}^{m-1} h_{k+1}(t)\,\mathbf{P}_k\,, \tag{3.30a}$$

where $\mathbf{P}_k = p_k(\mathbf{A})$ and the polynomials $p_k(z)$ are defined by (3.29b). The matrices $\mathbf{P}_k$ can be computed efficiently by the recipe

$$\begin{aligned} &\mathbf{P}_0 = \mathbf{I}\,, \\ &\mathbf{P}_1 = \mathbf{A} - z_1\mathbf{I}\,, \\ &\mathbf{P}_k = \mathbf{P}_{k-1}(\mathbf{A} - z_k\mathbf{I}) \quad \text{for } k = 2, \cdots, m-1\,. \end{aligned} \tag{3.30b}$$

This approach requires only $m-2$ matrix multiplications to compute the $\mathbf{P}_k$. Evaluating formula (3.30) when $m = n$ therefore requires $n^4 - 2n^3$ multiplications of numbers. This is 25 for $n = 3$ and 128 for $n = 4$ — much better!

Remark. This method is closely related to the Putzer Method for computing $e^{t\mathbf{A}}$. That method also uses formula (3.30) but rather than computing $h_1(t)$, $h_2(t)$, $\cdots$, $h_m(t)$ by using a divided-difference table, it obtains them by solving the system of differential equations

$$\begin{aligned} h_1' &= z_1 h_1 & h_1(0) &= 1\,, \\ h_k' &= z_k h_k + h_{k-1} & h_k(0) &= 0 \quad \text{for } k = 2, \cdots, m\,. \end{aligned}$$

Of course, the resulting $h_0(t)$, $h_0(t)$, $\cdots$, $h_{m-1}(t)$ are the same, but the divided-difference table gets to them faster. Moreover, the divided-difference table easily extends to other functions of matrices, whereas the Putzer Method does not.

For matrices annihilated by a quadratic polynonial, formula (3.30) does not require any matrix multiplications and it recovers formulas (3.15–3.17) for matrix exponentials that we derived earlier. There are two cases to consider.

Fact. If $\mathbf{A}$ is annihilated by a quadratic polynonial with a double root λ_1 then

$$e^{t\mathbf{A}} = e^{\lambda_1 t}\mathbf{I} + t\,e^{\lambda_1 t}(\mathbf{A} - \lambda_1\mathbf{I})\,.$$

Reason. Set $z_1 = z_2 = \lambda_1$. Then by (3.30b)

$$\mathbf{P}_0 = \mathbf{I}\,, \qquad \mathbf{P}_1 = \mathbf{A} - \lambda_1\mathbf{I}\,,$$

while by (3.29c) the divided-difference table is simply

$$\begin{array}{cccc} \lambda_1 & e^{\lambda_1 t} & & \\ & & \searrow & \\ \lambda_1 & e^{\lambda_1 t} & \rightarrow & t\,e^{\lambda_1 t} \end{array}$$

We read off from the top entries of the columns that

$$h_1(t) = e^{\lambda_1 t}\,, \qquad h_2(t) = t\,e^{\lambda_1 t}\,,$$

whereby the result follows from (3.30a).

Remark. The above fact recovers formula (3.17) for the matrix exponential when the quadratic polynonial $p(z)$ has a double root λ_1.

Fact. If $\mathbf{A}$ is annihilated by a quadratic polynonial with distinct roots λ_1 and λ_2 then

$$e^{t\mathbf{A}} = e^{\lambda_1 t}\mathbf{I} + \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}(\mathbf{A} - \lambda_1 \mathbf{I})\,.$$

Reason. Set $z_1 = \lambda_1$ and $z_2 = \lambda_2$. Then by (3.30b)

$$\mathbf{P}_0 = \mathbf{I}\,, \qquad \mathbf{P}_1 = \mathbf{A} - \lambda_1 \mathbf{I}\,,$$

while by (3.29c) the divided-difference table is simply

$$\begin{array}{cccc} \lambda_1 & e^{\lambda_1 t} & & \\ & & \searrow & \\ \lambda_2 & e^{\lambda_2 t} & \rightarrow & \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \end{array}$$

We read off from the top entries of the columns that

$$h_1(t) = e^{\lambda_1 t}\,, \qquad h_2(t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}\,,$$

whereby the result follows from (3.30a).

Remark. The above fact recovers formulas (3.15) and (3.16) for the matrix exponential when the quadratic polynonial $p(z)$ has distinct roots λ_1 and λ_2. For example, if λ_1 and λ_2 are real with $\lambda_1 = \mu - \nu$ and $\lambda_2 = \mu + \nu$ then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{(\mu-\nu)t}\mathbf{I} + \frac{e^{(\mu+\nu)t} - e^{(\mu-\nu)t}}{2\nu}\,(\mathbf{A} - (\mu-\nu)\mathbf{I}) \\ &= \frac{e^{(\mu+\nu)t} + e^{(\mu-\nu)t}}{2}\,\mathbf{I} + \frac{e^{(\mu+\nu)t} - e^{(\mu-\nu)t}}{2\nu}\,(\mathbf{A} - \mu\mathbf{I}) \\ &= e^{\mu t}\left[\cosh(\nu t)\,\mathbf{I} + \frac{\sinh(\nu t)}{\nu}\,(\mathbf{A} - \mu\mathbf{I})\right]\,. \end{aligned}$$

Similarly, if λ_1 and λ_2 are a conjugate pair with $\lambda_1 = \mu - i\nu$ and $\lambda_2 = \mu + i\nu$ then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{(\mu-i\nu)t}\mathbf{I} + \frac{e^{(\mu+i\nu)t} - e^{(\mu-i\nu)t}}{2i\nu}\,(\mathbf{A} - (\mu - i\nu)\mathbf{I}) \\ &= \frac{e^{(\mu+i\nu)t} + e^{(\mu-i\nu)t}}{2}\,\mathbf{I} + \frac{e^{(\mu+i\nu)t} - e^{(\mu-i\nu)t}}{2i\nu}\,(\mathbf{A} - \mu\mathbf{I}) \\ &= e^{\mu t}\left[\cos(\nu t)\,\mathbf{I} + \frac{\sin(\nu t)}{\nu}\,(\mathbf{A} - \mu\mathbf{I})\right] . \end{aligned}$$

It is clear that if λ_1 and λ_2 are a conjugate pair then computing $e^{t\mathbf{A}}$ by this approach is usually slower than simply applying formula (3.16) because of the complex arithmetic that is introduced.

For matrices annihilated by a cubic polynonial formula (3.30) requires only one matrix multiplication. There are three cases to consider.

Fact. If $\mathbf{A}$ is annihilated by a cubic polynonial with a triple root λ_1 then

$$e^{t\mathbf{A}} = e^{\lambda_1 t}\mathbf{I} + t\,e^{\lambda_1 t}(\mathbf{A} - \lambda_1\mathbf{I}) + \tfrac{1}{2}t^2 e^{\lambda_1 t}(\mathbf{A} - \lambda_1\mathbf{I})^2 .$$

Reason. Set $z_1 = z_2 = z_3 = \lambda_1$. Then by (3.30b)

$$\mathbf{P}_0 = \mathbf{I}\,, \qquad \mathbf{P}_1 = \mathbf{A} - \lambda_1\mathbf{I}\,, \qquad \mathbf{P}_2 = (\mathbf{A} - \lambda_1\mathbf{I})^2\,,$$

while by (3.29c) the divided-difference table is simply

$$\begin{array}{ccccccc} \lambda_1 & e^{\lambda_1 t} & & & & & \\ & & \searrow & & & & \\ \lambda_1 & e^{\lambda_1 t} & \rightarrow & t\,e^{\lambda_1 t} & & & \\ & & \searrow & & \searrow & & \\ \lambda_1 & e^{\lambda_1 t} & \rightarrow & t\,e^{\lambda_1 t} & \rightarrow & \frac{1}{2}t^2 e^{\lambda_1 t} \end{array}$$

We read off from the top entries of the columns that

$$h_1(t) = e^{\lambda_1 t}\,, \qquad h_2(t) = t\,e^{\lambda_1 t}\,, \qquad h_3(t) = \tfrac{1}{2}t^2 e^{\lambda_1 t}\,,$$

whereby the result follows from (3.30a).

Fact. If $\mathbf{A}$ is annihilated by a cubic polynonial with a double root λ_1 and a simple root λ_2 then

$$e^{t\mathbf{A}} = e^{\lambda_1 t}\mathbf{I} + t\,e^{\lambda_1 t}(\mathbf{A} - \lambda_1\mathbf{I}) + \frac{1}{\lambda_2 - \lambda_1}\left(\frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} - t\,e^{\lambda_1 t}\right)(\mathbf{A} - \lambda_1\mathbf{I})^2 .$$

Reason. Set $z_1 = z_2 = \lambda_1$ and $z_3 = \lambda_2$. Then by (3.30b)

$$\mathbf{P}_0 = \mathbf{I}\,, \qquad \mathbf{P}_1 = \mathbf{A} - \lambda_1\mathbf{I}\,, \qquad \mathbf{P}_2 = (\mathbf{A} - \lambda_1\mathbf{I})^2\,,$$

while by (3.29c) the divided-difference table is simply

$$\begin{array}{ccccccc} \lambda_1 & e^{\lambda_1 t} & & & & & \\ & & \searrow & & & & \\ \lambda_1 & e^{\lambda_1 t} & \rightarrow & t\,e^{\lambda_1 t} & & & \\ & & \searrow & & \searrow & & \\ \lambda_2 & e^{\lambda_2 t} & \rightarrow & \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} & \rightarrow & \dfrac{1}{\lambda_2 - \lambda_1}\left(\dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} - t\,e^{\lambda_1 t}\right) \end{array}$$

We read off from the top entries of the columns that

$$h_1(t) = e^{\lambda_1 t}\,, \qquad h_2(t) = t\, e^{\lambda_1 t}\,, \qquad h_3(t) = \frac{1}{\lambda_2 - \lambda_1}\left(\frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} - t\, e^{\lambda_1 t}\right),$$

whereby the result follows from (3.30a).

Fact. If $\mathbf{A}$ is annihilated by a cubic polynonial with distinct roots λ_1, λ_2, and λ_3 then

$$\begin{aligned} e^{t\mathbf{A}} = e^{\lambda_1 t}\mathbf{I} &+ \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}(\mathbf{A} - \lambda_1 \mathbf{I}) \\ &+ \frac{1}{\lambda_3 - \lambda_1}\left(\frac{e^{\lambda_3 t} - e^{\lambda_2 t}}{\lambda_3 - \lambda_2} - \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}\right)(\mathbf{A} - \lambda_1 \mathbf{I})(\mathbf{A} - \lambda_2 \mathbf{I})\,. \end{aligned}$$

Reason. Set $z_1 = \lambda_1$, $z_2 = \lambda_2$, and $z_3 = \lambda_3$. Then by (3.30b)

$$\mathbf{P}_0 = \mathbf{I}\,, \qquad \mathbf{P}_1 = \mathbf{A} - \lambda_1 \mathbf{I}\,, \qquad \mathbf{P}_2 = (\mathbf{A} - \lambda_1 \mathbf{I})(\mathbf{A} - \lambda_2 \mathbf{I})\,,$$

while by (3.29c) the divided-difference table is simply

$$\begin{array}{llllll} \lambda_1 & e^{\lambda_1 t} & & & & \\ & & \searrow & & & \\ \lambda_2 & e^{\lambda_2 t} & \rightarrow & \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} & & \\ & & \searrow & & \searrow & \\ \lambda_3 & e^{\lambda_3 t} & \rightarrow & \dfrac{e^{\lambda_3 t} - e^{\lambda_2 t}}{\lambda_3 - \lambda_2} & \rightarrow & \dfrac{1}{\lambda_3 - \lambda_1}\left(\dfrac{e^{\lambda_3 t} - e^{\lambda_2 t}}{\lambda_3 - \lambda_2} - \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}\right) \end{array}$$

We read off from the top entries of the columns that

$$h_1(t) = e^{\lambda_1 t}\,, \qquad h_2(t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}\,, \qquad h_3(t) = \frac{1}{\lambda_3 - \lambda_1}\left(\frac{e^{\lambda_3 t} - e^{\lambda_2 t}}{\lambda_3 - \lambda_2} - \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}\right),$$

whereby the result follows from (3.30a).

3.6.3. *Application to Other Functions of Matrices.* We can use the same method to compute other functions of matrices. Let $\mathbf{A}$ be an $n \times n$ real matrix and $p(z)$ be a polyinomial of degree $m \leq n$ that annihilates $\mathbf{A}$. Let $\{z_1, \cdots, z_m\}$ be the roots of $p(z)$ listed with multiplicity and clustered so that

$$\text{if } z_j = z_k \text{ for some } j < k \text{ then } z_j = z_{j+1} = \cdots = z_{k-1} = z_k\,.$$

Let $f(z)$ be a function that is defined and differentiable at every complex z. (Such functions are called *entire*.) For example, let $f(z) = z^r$ for some positive integer r, where we think of r as large. Compute the divided-difference table whose first column is seeded with $h[z_k] = f(z_k)$ and subsequent columns are given by (3.29c). Read off the coefficients h_1, h_2, $\cdots$, h_m from the top entries of each column as prescribed by (3.29d). Then set

$$f(\mathbf{A}) = h(\mathbf{A}) = \sum_{k=0}^{m-1} h_{k+1}\, \mathbf{P}_k\,, \tag{3.31}$$

where the matices $\mathbf{P}_k$ are computed by (3.30b). In this way we can compute $\mathbf{A}^{1000}$ efficiently when n is not too large.

4. Eigen Methods

4.1. **Eigenpairs.** Let $\mathbf{A}$ be a real $n \times n$ matrix. A number λ (possibly complex) is an *eigenvalue* of $\mathbf{A}$ if there exists a nonzero vector $\mathbf{v}$ (possibly complex) such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} . \tag{4.1}$$

Each such vector is an *eigenvector* associated with λ, and $(\lambda, \mathbf{v})$ is an *eigenpair* of $\mathbf{A}$.

Fact 1. If $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$ then so is $(\lambda, \alpha\mathbf{v})$ for every complex $\alpha \neq 0$. In other words, if $\mathbf{v}$ is an eigenvector associated with an eigenvalue λ of $\mathbf{A}$ then so is $\alpha\mathbf{v}$ for every complex $\alpha \neq 0$. In particular, eigenvectors are not unique.

Reason. Because $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$ we know that (4.1) holds. It follows that

$$\mathbf{A}(\alpha\mathbf{v}) = \alpha\mathbf{A}\mathbf{v} = \alpha\lambda\mathbf{v} = \lambda(\alpha\mathbf{v}) .$$

Because the scalar α and vector $\mathbf{v}$ are nonzero, the vector $\alpha\mathbf{v}$ is also nonzero. Therefore $(\lambda, \alpha\mathbf{v})$ is also an eigenpair of $\mathbf{A}$. □

4.1.1. *Finding Eigenvalues.* Recall that the characteristic polynomial of $\mathbf{A}$ is defined by

$$p_{\mathbf{A}}(z) = \det(z\mathbf{I} - \mathbf{A}) .$$

It has the form

$$p_{\mathbf{A}}(z) = z^n + \pi_1 z^{n-1} + \pi_2 z^{n-2} + \cdots + \pi_{n-1} z + \pi_n ,$$

where the coefficients π_1, π_2, $\cdots$, π_n are real. In other words, it is a real monic polynomial of degree n. One can show that in general

$$\pi_1 = -\operatorname{tr}(\mathbf{A}) , \qquad \pi_n = (-1)^n \det(\mathbf{A}) ,$$

where $\operatorname{tr}(\mathbf{A})$ is the sum of the diagonal entries of $\mathbf{A}$. In particular, when $n = 2$ one has

$$p_{\mathbf{A}}(z) = z^2 - \operatorname{tr}(\mathbf{A}) z + \det(\mathbf{A}) .$$

Because $\det(z\mathbf{I} - \mathbf{A}) = (-1)^n \det(\mathbf{A} - z\mathbf{I})$, this definition of $p_{\mathbf{A}}(z)$ coincides with the book's definition when n is even, and is its negative when n is odd. Both conventions are common. We have chosen the convention that makes $p_{\mathbf{A}}(z)$ monic. What matters most about $p_{\mathbf{A}}(z)$ is its roots and their multiplicity, which are the same for both conventions.

Fact 2. A number λ is an eigenvalue of $\mathbf{A}$ if and only if $p_{\mathbf{A}}(\lambda) = 0$. In other words, the eigenvalues of $\mathbf{A}$ are the roots of $p_{\mathbf{A}}(z)$.

Reason. If λ is an eigenvalue of $\mathbf{A}$ then by (4.1) there exists a nonzero vector $\mathbf{v}$ such that

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \lambda\mathbf{v} - \mathbf{A}\mathbf{v} = \mathbf{0} .$$

It follows that $p_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = 0$.

Conversely, if $p_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then there exists a nonzero vector $\mathbf{v}$ such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$. It follows that

$$\lambda\mathbf{v} - \mathbf{A}\mathbf{v} = (\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0} ,$$

whereby λ and $\mathbf{v}$ satisfy (4.1), which implies λ is an eigenvalue of $\mathbf{A}$. □

Fact 2 shows that the eigenvalues of a $n \times n$ matrix $\mathbf{A}$ can be found if we can find all the roots of the characteristic polynomial of $\mathbf{A}$. Because the degree of this characteristic polynomial is n, and because every polynomical of degree n has exactly n roots counting multiplicity, the $n\times n$ matrix $\mathbf{A}$ therefore must have at least one eigenvalue and can have at most n eigenvalues.

Example. Find the eigenvalues of $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{A}}(z) = z^2 - 6z + 5 = (z-1)(z-5)\,.$$

By Fact 2 the eigenvalues of $\mathbf{A}$ are 1 and 5.

Example. Find the eigenvalues of $\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{A}}(z) = z^2 - 6z + 9 = (z-3)^2\,.$$

By Fact 2 the only eigenvalue of $\mathbf{A}$ is 3.

Example. Find the eigenvalues of $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{A}}(z) = z^2 - 6z + 13 = (z-3)^2 + 4 = (z-3)^2 + 2^2\,.$$

By Fact 2 the eigenvalues of $\mathbf{A}$ are $3 + i2$ and $3 - i2$.

Remark. The above examples show all the possibilities that can arise for 2×2 matrices — namely, a 2×2 matrix $\mathbf{A}$ can have either, two real eigenvalues, one real eigenvalue, or a conjugate pair of eigenvalues.

4.1.2. *Finding Eigenvectors for* 2×2 *Matrices.* Once we have found the eigenvalues of a matrix, we can find all the eigenvectors associated with each eigenvalue by finding a general nonzero solution of (4.1). For an $n\times n$ matrix with n distinct eigenvalues this means solving n homogeneous linear algebraic systems, which might take some time. However, there is a trick that allows us to quickly find all the eigenvectors associated with each eigenvalue of any 2×2 matrix $\mathbf{A}$ without solving any linear systems.

The trick is based on the Cayley-Hamilton Theorem, which states that $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$. The eigenvalues λ_1 and λ_2 are the roots of $p_{\mathbf{A}}(z)$, so that $p_{\mathbf{A}}(z) = (z-\lambda_1)(z-\lambda_2)$. Hence, by the Cayley-Hamilton Theorem

$$\mathbf{0} = p_{\mathbf{A}}(\mathbf{A}) = (\mathbf{A} - \lambda_1\mathbf{I})(\mathbf{A} - \lambda_2\mathbf{I}) = (\mathbf{A} - \lambda_2\mathbf{I})(\mathbf{A} - \lambda_1\mathbf{I})\,. \tag{4.2}$$

It follows that every nonzero column of $\mathbf{A} - \lambda_2\mathbf{I}$ is an eigenvector associated with λ_1, and that every nonzero column of $\mathbf{A} - \lambda_1\mathbf{I}$ is an eigenvector associated with λ_2. This observation also applies to the case where $\lambda_1 = \lambda_2$ and $\mathbf{A} - \lambda_1\mathbf{I} \neq \mathbf{0}$. Of course, if $\mathbf{A} - \lambda_1\mathbf{I} = \mathbf{0}$ then every nonzero vector is an eigenvector.

Example. Find an eigenpair for each of the eigenvalues of $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution. We have already showed that the eigenvalues of $\mathbf{A}$ are 1 and 5. Compute

$$\mathbf{A}-\mathbf{I}=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}, \qquad \mathbf{A}-5\mathbf{I}=\begin{pmatrix}-2 & 2\\ 2 & -2\end{pmatrix},$$

Every column of $\mathbf{A}-5\mathbf{I}$ has the form

$$\alpha\begin{pmatrix}1\\ -1\end{pmatrix} \quad \text{for some } \alpha \neq 0\,,$$

while every column of $\mathbf{A}-\mathbf{I}$ has the form

$$\beta\begin{pmatrix}1\\ 1\end{pmatrix} \quad \text{for some } \beta \neq 0\,.$$

It follows from (4.2) that eigenpairs of $\mathbf{A}$ are

$$\left(1\,,\begin{pmatrix}1\\ -1\end{pmatrix}\right), \qquad \left(5\,,\begin{pmatrix}1\\ 1\end{pmatrix}\right).$$

Example. Find an eigenpair for each of the eigenvalues of $\mathbf{A}=\begin{pmatrix}4 & -1\\ 1 & 2\end{pmatrix}$.

Solution. We have already showed that the only eigenvalue of $\mathbf{A}$ is 3. Compute

$$\mathbf{A}-3\mathbf{I}=\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}.$$

Every column of $\mathbf{A}-3\mathbf{I}$ has the form

$$\alpha\begin{pmatrix}1\\ 1\end{pmatrix} \quad \text{for some } \alpha \neq 0\,.$$

It follows from (4.2) that an eigenpair of $\mathbf{A}$ is

$$\left(3\,,\begin{pmatrix}1\\ 1\end{pmatrix}\right).$$

Example. Find an eigenpair for each of the eigenvalues of $\mathbf{A}=\begin{pmatrix}3 & 2\\ -2 & 3\end{pmatrix}$.

Solution. We have already showed that the eigenvalues of $\mathbf{A}$ are $3+i2$ and $3-i2$. Compute

$$\mathbf{A}-(3+i2)\mathbf{I}=\begin{pmatrix}-i2 & 2\\ -2 & -i2\end{pmatrix}, \qquad \mathbf{A}-(3-i2)\mathbf{I}=\begin{pmatrix}i2 & 2\\ -2 & i2\end{pmatrix}.$$

Every column of $\mathbf{A}-(3-i2)\mathbf{I}$ has the form

$$\alpha\begin{pmatrix}1\\ i\end{pmatrix} \quad \text{for some } \alpha \neq 0\,,$$

while every column of $\mathbf{A}-(3+i2)\mathbf{I}$ has the form

$$\beta\begin{pmatrix}1\\ -i\end{pmatrix} \quad \text{for some } \beta \neq 0\,.$$

It follows from (4.2) that eigenpairs of $\mathbf{A}$ are

$$\left(3+i2\,,\begin{pmatrix}1\\ i\end{pmatrix}\right)\,,\qquad \left(3-i2\,,\begin{pmatrix}1\\ -i\end{pmatrix}\right)\,.$$

Notice that in the above example the eigenvectors associated with $3-i2$ are complex conjugates to those associated with $3+i2$. This illustrates a particular instance of the following general fact.

Fact 3. If $(\lambda,\mathbf{v})$ is an eigenpair of the real matrix $\mathbf{A}$ then so is $(\overline{\lambda},\overline{\mathbf{v}})$.

Reason. Because $(\lambda,\mathbf{v})$ is an eigenpair of $\mathbf{A}$ we know by (4.1) that $\mathbf{A}\mathbf{v}=\lambda\mathbf{v}$. Because $\mathbf{A}$ is real, the complex conjugate of this equation is

$$\mathbf{A}\overline{\mathbf{v}}=\overline{\lambda}\overline{\mathbf{v}}\,,$$

where $\overline{\mathbf{v}}$ is nonzero because $\mathbf{v}$ is nonzero. It follows that $(\overline{\lambda},\overline{\mathbf{v}})$ is an eigenpair of $\mathbf{A}$. □

The foregoing examples illustrate particular instances of the following general facts.

Fact 4. Let λ be an eigenvalue of the real matrix $\mathbf{A}$. If λ is real then it has a real eigenvector. If λ is not real then none of its eigenvectors are real.

Reason. Let $\mathbf{v}$ be any eigenvector associated with λ, so that $(\lambda,\mathbf{v})$ is an eigenpair of $\mathbf{A}$. Let $\lambda=\mu+i\nu$ and $\mathbf{v}=\mathbf{u}+i\mathbf{w}$ where μ and ν are real numbers and $\mathbf{u}$ and $\mathbf{w}$ are real vectors. One then has

$$\mathbf{A}\mathbf{u}+i\mathbf{A}\mathbf{w}=\mathbf{A}\mathbf{v}=\lambda\mathbf{v}=(\mu+i\nu)(\mathbf{u}+i\mathbf{w})=(\mu\mathbf{u}-\nu\mathbf{w})+i(\mu\mathbf{w}+\nu\mathbf{u})\,,$$

which is equivalent to

$$\mathbf{A}\mathbf{u}-\mu\mathbf{u}=-\nu\mathbf{w}\,,\qquad\text{and}\qquad \mathbf{A}\mathbf{w}-\mu\mathbf{w}=\nu\mathbf{u}\,.$$

If $\nu=0$ then $\mathbf{u}$ and $\mathbf{w}$ will be real eigenvectors associated with λ whenever they are nonzero. But at least one of $\mathbf{u}$ and $\mathbf{w}$ must be nonzero because $\mathbf{v}=\mathbf{u}+i\mathbf{w}$ is nonzero. Conversely, if $\nu\neq 0$ and $\mathbf{w}=\mathbf{0}$ then the second equation above implies $\mathbf{u}=\mathbf{0}$ too, which contradicts the fact that at least one of $\mathbf{u}$ and $\mathbf{w}$ must be nonzero. Hence, if $\nu\neq 0$ then $\mathbf{w}\neq\mathbf{0}$. □

4.2. **Special Solutions of First-Order Systems.** Eigenvalues and eigenvectors can be used to construct special solutions of first-order differential systems with a constant coefficient matrix. The system we study is

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}=\mathbf{A}\mathbf{x}\,,\tag{4.3}$$

where $\mathbf{x}(t)$ is a vector and $\mathbf{A}$ is a real $n{\times}n$ matrix. We begin with the following fact.

Fact 5. If $(\lambda,\mathbf{v})$ is an eigenpair of $\mathbf{A}$ then a solution of system (4.3) is

$$\mathbf{x}(t)=e^{\lambda t}\mathbf{v}\,.\tag{4.4}$$

Reason. Because $\lambda\mathbf{v}=\mathbf{A}\mathbf{v}$, a direct calculation shows that

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{\lambda t}\mathbf{v}\big)=e^{\lambda t}\lambda\mathbf{v}=e^{\lambda t}\mathbf{A}\mathbf{v}=\mathbf{A}\big(e^{\lambda t}\mathbf{v}\big)=\mathbf{A}\mathbf{x}\,,$$

whereby $\mathbf{x}(t)$ given by (4.4) solves (4.3). □

If $(\lambda, \mathbf{v})$ is a real eigenpair of $\mathbf{A}$ then recipe (4.4) will yield a real solution of (4.3). But if λ is an eigenvalue of $\mathbf{A}$ that is not real then recipe (4.4) will not yield a real solution. However, if we also use the solution associated with the conjugate eigenpair $(\overline{\lambda}, \overline{\mathbf{v}})$ then we can construct two real solutions.

Fact 6. Let $(\lambda, \mathbf{v})$ be an eigenpair of $\mathbf{A}$ with $\lambda = \mu + i\nu$ and $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ where μ and ν are real numbers while $\mathbf{u}$ and $\mathbf{w}$ are real vectors. Then two real solutions of system (4.3) are

$$\begin{aligned} \mathbf{x}_1(t) &= \mathrm{Re}\big(e^{\lambda t}\mathbf{v}\big) = e^{\mu t}\big(\mathbf{u}\cos(\nu t) - \mathbf{w}\sin(\nu t)\big)\,, \\ \mathbf{x}_2(t) &= \mathrm{Im}\big(e^{\lambda t}\mathbf{v}\big) = e^{\mu t}\big(\mathbf{w}\cos(\nu t) + \mathbf{u}\sin(\nu t)\big)\,. \end{aligned} \tag{4.5}$$

Reason. Because $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$, by Fact 3 so is $(\overline{\lambda}, \overline{\mathbf{v}})$. By recipe (4.4) two solutions of (4.3) are $e^{\lambda t}\mathbf{v}$ and $e^{\overline{\lambda}t}\overline{\mathbf{v}}$, which are complex conjugates of each other. Because system (4.3) is linear, it follows that two real solutions of (4.3) are given by

$$\mathbf{x}_1(t) = \mathrm{Re}\big(e^{\lambda t}\mathbf{v}\big) = \frac{e^{\lambda t}\mathbf{v} + e^{\overline{\lambda}t}\overline{\mathbf{v}}}{2}\,, \qquad \mathbf{x}_2(t) = \mathrm{Im}\big(e^{\lambda t}\mathbf{v}\big) = \frac{e^{\lambda t}\mathbf{v} - e^{\overline{\lambda}t}\overline{\mathbf{v}}}{i2}\,.$$

Because $\lambda = \mu + i\nu$ and $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ we see that

$$\begin{aligned} e^{\lambda t}\mathbf{v} &= e^{\mu t}\big(\cos(\nu t) + i\sin(\nu t)\big)(\mathbf{u} + i\mathbf{v}) \\ &= e^{\mu t}\Big[\big(\mathbf{u}\cos(\nu t) - \mathbf{w}\sin(\nu t)\big) + i\big(\mathbf{w}\cos(\nu t) + \mathbf{u}\sin(\nu t)\big)\Big]\,, \end{aligned}$$

whereby $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are the real and imaginary parts, yielding (4.5). □

Recall that if we have n linearly independent real solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$, of system (4.3) then we can construct a fundamental matrix $\mathbf{\Psi}(t)$ by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix}\,.$$

Recipes (4.4) and (4.5) will often, but not always, yield enough solutions to do this.

Example. Use eigenpairs to construct real solutions of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}\,, \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\,.$$

If possible, use these solutions to construct a fundamental matrix $\mathbf{\Psi}(t)$ for this system.

Solution. By a previous example we know that $\mathbf{A}$ has the real eigenpairs

$$\left(1\,, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\,, \qquad \left(5\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)\,.$$

By recipe (4.4) the system has the real solutions

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,, \qquad \mathbf{x}_2(t) = e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,.$$

These solutions are linearly independent because

$$W[\mathbf{x}_1, \mathbf{x}_2](0) = \det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2 \neq 0\,.$$

Therefore a fundamental matrix for the system is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} e^t & e^{5t} \\ -e^t & e^{5t} \end{pmatrix} .$$

Example. Use eigenpairs to construct real solutions of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} , \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} .$$

If possible, use these solutions to construct a fundamental matrix $\mathbf{\Psi}(t)$ for this system.

Solution. By a previous example we know that $\mathbf{A}$ has the eigenpair

$$\left(3 , \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) .$$

By recipe (4.4) the system has the real solution

$$\mathbf{x}(t) = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Because we have not constructed two linearly independent solutions, we cannot yet construct a fundamental matrix for this system.

Example. Use eigenpairs to construct real solutions of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} , \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} .$$

If possible, use these solutions to construct a fundamental matrix $\mathbf{\Psi}(t)$ for this system.

Solution. By a previous example we know that $\mathbf{A}$ has the conjugate eigenpairs

$$\left(3 + i2 , \begin{pmatrix} 1 \\ i \end{pmatrix} \right) , \qquad \left(3 - i2 , \begin{pmatrix} 1 \\ -i \end{pmatrix} \right) .$$

Because

$$e^{(3+i2)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{3t} \big(\cos(2t) + i \sin(2t) \big) \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{3t} \begin{pmatrix} \cos(2t) + i \sin(2t) \\ -\sin(2t) + i \cos(2t) \end{pmatrix} ,$$

by recipe (4.5) the system has the real solutions

$$\mathbf{x}_1(t) = e^{3t} \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} , \qquad \mathbf{x}_2(t) = e^{3t} \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} .$$

These solutions are linearly independent because

$$W[\mathbf{x}_1, \mathbf{x}_2](0) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0 .$$

Therefore a fundamental matrix for the system is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = e^{3t} \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} .$$

Remark. Recipes (4.4) and (4.5) will always yield n linearly independent real solutions of system (4.3) whenever $p_{\mathbf{A}}(z)$ only has simple roots, as was the case in the first and third examples above. They will typically fail to do so whenever $p_{\mathbf{A}}(z)$ has a root with multiplicity greater than 1, as was the case in the second example above.

Finally, whenever we can construct a fundamental matrix $\mathbf{\Psi}(t)$ for system (4.3) from special solutions, we can compute $e^{t\mathbf{A}}$ by using the fact that

$$e^{t\mathbf{A}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} .$$

It is easy to check that the right-hand side above satisfies the matrix-valued initial-value problem (3.2), so it must be equal to $e^{t\mathbf{A}}$.

Example. Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution. In the first example given above we constructed a fundamental matrix $\mathbf{\Psi}(t)$ for the associated linear system. We obtained

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^t & e^{5t} \\ -e^t & e^{5t} \end{pmatrix} .$$

Because

$$\mathbf{\Psi}(0) = \begin{pmatrix} e^0 & e^0 \\ -e^0 & e^0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} ,$$

and $\det(\mathbf{\Psi}(0)) = 1 - (-1) = 2$, we see that

$$\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} ,$$

whereby

$$e^{t\mathbf{A}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} e^t & e^{5t} \\ -e^t & e^{5t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^t + e^{5t} & -e^t + e^{5t} \\ -e^t + e^{5t} & e^t + e^{5t} \end{pmatrix} .$$

Example. Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$.

Solution. In the third example given above we constructed a fundamental matrix $\mathbf{\Psi}(t)$ for the associated linear system. We obtained

$$\mathbf{\Psi}(t) = e^{3t} \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} .$$

Because

$$\mathbf{\Psi}(0) = e^0 \begin{pmatrix} \cos(0) & \sin(0) \\ -\sin(0) & \cos(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} ,$$

we see that $\mathbf{\Psi}(0)^{-1} = \mathbf{I}$, whereby

$$e^{t\mathbf{A}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} = \mathbf{\Psi}(t)\mathbf{I} = \mathbf{\Psi}(t) = e^{3t} \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} .$$

4.3. **Two-by-Two Fundamental Matrices.** Recipes (4.4) and (4.5) will always yield n linearly independent real solutions of system (4.3) with which we can construct a real fundamental matrix whenever the characteristic polynomial $p_{\mathbf{A}}(z)$ only has simple roots. For 2×2 matrices they will fail to do so whenever $p_{\mathbf{A}}(z) = (z-\mu)^2$ and $\mathbf{A} - \mu\mathbf{I} \neq \mathbf{0}$. In that case μ is the only eigenvalue of $\mathbf{A}$ and recipe (4.4) yields the real solution

$$\mathbf{x}_1(t) = e^{\mu t}\mathbf{v}\,,$$

where the vector $\mathbf{v}$ is proportional to any nonzero column of $\mathbf{A} - \mu\mathbf{I}$. Then if $\mathbf{w}$ is any vector that is *not* proportional to $\mathbf{v}$, we can construct a second solution by the recipe

$$\mathbf{x}_2(t) = e^{\mu t}\mathbf{w} + t\,e^{\mu t}(\mathbf{A} - \mu\mathbf{I})\mathbf{w}\,. \tag{4.6}$$

This is just $e^{t\mathbf{A}}\mathbf{w}$ where $e^{t\mathbf{A}}$ is given by (3.14). We can always take the vector $\mathbf{w}$ to be proportional to the transpose of any nonzero row of $\mathbf{A} - \mu\mathbf{I}$. We can then construct the real fundamental matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t)\end{pmatrix}\,.$$

This will yield a fundamental matrix only when $\mathbf{A}$ is 2×2.

Example. Construct a real fundamental matrix for the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}\,, \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix}4 & -1\\ 1 & 2\end{pmatrix}\,.$$

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{A}}(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 9 = (z-3)^2\,.$$

Because

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\,,$$

we see that $\mathbf{A}$ has the eigenpair

$$\left(3\,, \begin{pmatrix}1\\ 1\end{pmatrix}\right)\,.$$

Recipe (4.4) yields the real solution

$$\mathbf{x}_1(t) = e^{3t}\begin{pmatrix}1\\ 1\end{pmatrix}\,.$$

Recipe (4.6) with $\mathbf{w} = \begin{pmatrix}1 & -1\end{pmatrix}^T$ yields the second real solution

$$\begin{aligned}\mathbf{x}_2(t) &= e^{3t}\begin{pmatrix}1\\ -1\end{pmatrix} + t\,e^{3t}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}\\ &= e^{3t}\begin{pmatrix}1\\ -1\end{pmatrix} + t\,e^{3t}\begin{pmatrix}2\\ 2\end{pmatrix} = e^{3t}\begin{pmatrix}1+2t\\ -1+2t\end{pmatrix}\,.\end{aligned}$$

Therefore a real fundamental matrix for the system is given by

$$\mathbf{\Psi}(t) = e^{3t}\begin{pmatrix}1 & 1+2t\\ 1 & -1+2t\end{pmatrix}\,.$$

4.4. **Diagonalizable Matrices.** If recipe (4.4) yields n linearly independent solutions of the first-order system (4.3) then they can be used to directly construct $e^{t\mathbf{A}}$. The key to this construction is the following fact from linear algebra.

Fact 7. If a real $n\times n$ matrix $\mathbf{A}$ has n eigenpairs, $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $\cdots$, $(\lambda_n, \mathbf{v}_n)$, such that the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent then

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} \,, \tag{4.7}$$

where $\mathbf{V}$ is the $n\times n$ matrix whose columns are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ — i.e.

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} , \tag{4.8}$$

while $\mathbf{D}$ is the $n\times n$ diagonal matrix

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} . \tag{4.9}$$

Reason. Underlying this result is the fact that

$$\begin{aligned} \mathbf{A}\mathbf{V} &= \mathbf{A} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \cdots & \mathbf{A}\mathbf{v}_n \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} = \mathbf{V}\mathbf{D} \,. \end{aligned} \tag{4.10}$$

Once we show that $\mathbf{V}$ is invertible then (4.7) follows upon multiplying the above relation on the right by $\mathbf{V}^{-1}$.

We claim that $\det(\mathbf{V}) \neq 0$ because the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent. Suppose otherwise. Because $\det(\mathbf{V}) = 0$ there exists a nonzero vector $\mathbf{c}$ such that $\mathbf{V}\mathbf{c} = \mathbf{0}$. This means that

$$\mathbf{0} = \mathbf{V}\mathbf{c} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \,.$$

Because the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent, the above relation implies that $c_1 = c_2 = \cdots = c_n = 0$, which contradicts the fact $\mathbf{c}$ is nonzero. Therefore $\det(\mathbf{V}) \neq 0$. Hence, the matrix $\mathbf{V}$ is invertible and (4.7) follows upon multiplying relation (4.10) on the right by $\mathbf{V}^{-1}$. □

We call a real $n\times n$ matrix $\mathbf{A}$ *diagonalizable* when there exists an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$. To *diagonalize* $\mathbf{A}$ means to find such a $\mathbf{V}$ and $\mathbf{D}$. Fact 7 states that $\mathbf{A}$ is diagonalizable when it has n linearly independent eigenvectors. The converse of this statement is also true.

Fact 8. If a real $n \times n$ matrix $\mathbf{A}$ is diagonalizable then it has n linearly independent eigenvectors.

Reason. Because $\mathbf{A}$ is diagonalizable it has the form $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ where the matrix $\mathbf{V}$ is invertible and the matrix $\mathbf{D}$ is diagonal.

Let the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ be the columns of $\mathbf{V}$. We claim these vectors are linearly independent. Indeed, if $\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ then because $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$ we see that

$$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{V}\mathbf{c}\,.$$

Because $\mathbf{V}$ is invertible, this implies that $\mathbf{c} = \mathbf{0}$. The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ therefore are linearly independent.

Because $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$ and because $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ where $\mathbf{D}$ has the form (4.9), we see that

$$\begin{aligned}
\begin{pmatrix} \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \cdots & \mathbf{A}\mathbf{v}_n \end{pmatrix} &= \mathbf{A} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \\
&= \mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{V} = \mathbf{V}\mathbf{D} \\
&= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{pmatrix} .
\end{aligned}$$

Because the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent, they are all nonzero. It then follows from the above relation that $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $\cdots$, $(\lambda_n, \mathbf{v}_n)$ are eigenpairs of $\mathbf{A}$, such that the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent. □

Example. Show that $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ is diagonalizable, and diagonalize it.

Solution. By a previous example we know that $\mathbf{A}$ has the real eigenpairs

$$\left(1\,, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) , \qquad \left(5\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) .$$

Because we also know the eigenvectors are linearly independent, $\mathbf{A}$ is diagonalizable. Then (4.8) and (4.9) yield

$$\mathbf{V} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} , \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} .$$

Because $\det(\mathbf{V}) = 2$, one has

$$\mathbf{V}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .$$

It follows from (4.7) that $\mathbf{A}$ is diagonalized as

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Example. Show that $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$ is diagonalizable, and diagonalize it.

Solution. By a previous example we know that $\mathbf{A}$ has the conjugate eigenpairs

$$\left(3 + i2\,, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \qquad \left(3 - i2\,, \begin{pmatrix} 1 \\ -i \end{pmatrix}\right).$$

Because we also know the eigenvectors are linearly independent, $\mathbf{A}$ is diagonalizable. Then (4.8) and (4.9) yield

$$\mathbf{V} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 3+i2 & 0 \\ 0 & 3-i2 \end{pmatrix}.$$

Because $\det(\mathbf{V}) = -i2$, we have

$$\mathbf{V}^{-1} = \frac{1}{-i2}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

It follows from (4.7) that $\mathbf{A}$ is diagonalized as

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} 3+i2 & 0 \\ 0 & 3-i2 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Example. Use Fact 8 to show that $\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$ is not diagonalizable.

Solution. By previous examples we know that $\mathbf{A}$ only has one real eigenvalue 3 and that all eigenvectors associated with 3 have the form

$$\alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{for some } \alpha \neq 0\,.$$

Because $\mathbf{A}$ does not have two linearly independent eigenvectors, it is not diagonalizable.

Remark. While not every matrix is diagonalizable, most matrices are. Here we give four criteria that insure a real $n\times n$ matrix $\mathbf{A}$ is diagonalizable.

- If $\mathbf{A}$ has n distinct eigenvalues then it is diagonalizable.
- If $\mathbf{A}$ is symmetric ($\mathbf{A}^T = \mathbf{A}$) then all of its eigenvalues are real ($\overline{\lambda_j} = \lambda_j$), and it will have n real eigenvectors $\mathbf{v}_j$ that can be normalized so that $\mathbf{v}_j^T\mathbf{v}_k = \delta_{jk}$. With this normalization $\mathbf{V}^{-1} = \mathbf{V}^T$.
- If $\mathbf{A}$ is skew-symmetric ($\mathbf{A}^T = -\mathbf{A}$) then all of its eigenvalues are imaginary ($\overline{\lambda_j} = -\lambda_j$), and it will have n eigenvectors $\mathbf{v}_j$ that can be normalized so that $\mathbf{v}_j^*\mathbf{v}_k = \delta_{jk}$. With this normalization $\mathbf{V}^{-1} = \mathbf{V}^*$.
- If $\mathbf{A}$ is normal ($\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$) then it will have n eigenvectors $\mathbf{v}_j$ that can be normalized so that $\mathbf{v}_j^*\mathbf{v}_k = \delta_{jk}$. With this normalization $\mathbf{V}^{-1} = \mathbf{V}^*$.

Matrices that are either symmetric or skew-symmetric are also normal. There are normal matrices that are neither symmetric nor skew-symmetric. Because the normal criterion is harder to verify than the symmetric and skew-symmetric criteria, it should be checked last. The first two examples that we gave above have distinct eigenvalues. The first example is symmetric. The second is normal, but is neither symmetric nor skew-symmetric.

4.5. **Computing Matrix Exponentials by Diagonalization.** We are now ready to give a direct construction of the matrix exponential $e^{t\mathbf{A}}$ when $\mathbf{A}$ is diagonalizable.

Fact 9. If the real $n\times n$ matrix $\mathbf{A}$ has n eigenpairs, $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $\cdots$, $(\lambda_n, \mathbf{v}_n)$, such that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent then

$$e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}\,, \tag{4.11}$$

where $\mathbf{V}$ and $\mathbf{D}$ are the $n\times n$ matrices given by (4.8) and (4.9).

Reason. Set $\mathbf{\Phi}(t) = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. It then follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Phi}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}\big) = \mathbf{V}\frac{\mathrm{d}}{\mathrm{d}t}e^{t\mathbf{D}}\mathbf{V}^{-1} = \mathbf{V}\mathbf{D}e^{t\mathbf{D}}\mathbf{V}^{-1} = \mathbf{A}\mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} = \mathbf{A}\mathbf{\Phi}(t)\,,$$

whereby the matrix-valued function $\mathbf{\Phi}(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Phi}(t) = \mathbf{A}\mathbf{\Phi}(t)\,.$$

Moreover, because $e^{0\mathbf{D}} = \mathbf{I}$ we see that $\mathbf{\Phi}(t)$ also satisfies the initial condition

$$\mathbf{\Phi}(0) = \mathbf{V}e^{0\mathbf{D}}\mathbf{V}^{-1} = \mathbf{V}\mathbf{I}\mathbf{V}^{-1} = \mathbf{V}\mathbf{V}^{-1} = \mathbf{I}\,.$$

Therefore $\mathbf{\Phi}(t)$ satisfies the matrix-valued initial-value problem (3.2), which implies $\mathbf{\Phi}(t) = e^{t\mathbf{A}}$, whereby (4.11) follows. $\square$

Formula (4.11) is the textbook's method for computing $e^{t\mathbf{A}}$ when $\mathbf{A}$ is diagonalizable. Because not every matrix is diagonalizable, it cannot always be applied. When it can be applied, most of the work needed to apply it goes into computing $\mathbf{V}$ and $\mathbf{V}^{-1}$. The matrix $e^{t\mathbf{D}}$ is simply given by

$$e^{t\mathbf{D}} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{pmatrix}. \tag{4.12}$$

Once we have $\mathbf{V}$, $\mathbf{V}^{-1}$, and $e^{t\mathbf{D}}$, formula (4.11) requires two matrix multiplications.

Example. Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution. By a previous example we know that $\mathbf{A}$ is diagonalizable with $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ where

$$\mathbf{V} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \qquad \mathbf{V}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

By formulas (4.11) and (4.12) we have

$$
\begin{aligned}
e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^t & -e^t \\ e^{5t} & e^{5t} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5t}+e^t & e^{5t}-e^t \\ e^{5t}-e^t & e^{5t}+e^t \end{pmatrix} .
\end{aligned}
$$

Example. Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$.

Solution. By a previous example we know that $\mathbf{A}$ is diagonalizable with $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ where

$$
\mathbf{V} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} , \qquad \mathbf{D} = \begin{pmatrix} 3+i2 & 0 \\ 0 & 3-i2 \end{pmatrix} , \qquad \mathbf{V}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} .
$$

By formula (4.12) we have

$$
e^{t\mathbf{D}} = \begin{pmatrix} e^{(3+i2)t} & 0 \\ 0 & e^{(3-i2)t} \end{pmatrix} = e^{3t}\begin{pmatrix} e^{i2t} & 0 \\ 0 & e^{-i2t} \end{pmatrix} .
$$

By formula (4.11) we have

$$
\begin{aligned}
e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} &= \frac{e^{3t}}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{i2t} & 0 \\ 0 & e^{-i2t} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\
&= \frac{e^{3t}}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{i2t} & -ie^{i2t} \\ e^{-i2t} & ie^{-i2t} \end{pmatrix} = \frac{e^{3t}}{2}\begin{pmatrix} e^{i2t}+e^{-i2t} & -ie^{i2t}+ie^{-i2t} \\ ie^{i2t}-ie^{-i2t} & e^{i2t}+e^{-i2t} \end{pmatrix} \\
&= \frac{e^{3t}}{2}\begin{pmatrix} 2\cos(2t) & 2\sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} = e^{3t}\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} .
\end{aligned}
$$

Remark. Because $\mathbf{A}$ is real, $e^{t\mathbf{A}}$ must be real. As the previous example illustrates, the matrices $\mathbf{V}$ and $\mathbf{D}$ may not be real, but will always combine in formula (4.11) to yield the real result. In particular, formula (4.11) allows us to directly compute $e^{t\mathbf{A}}$ without computing a fundamental set of real solutions as an intermediate step.

Remark. Formula (4.11) is a far more efficient way to compute $e^{t\mathbf{A}}$ than formula (3.10) whenever $\mathbf{A}$ is diagonalizable and is not annihilated by a polynonial of small degree, as is usually the case when n is large. This is because computing all the powers $\mathbf{A}$ needed in formula (3.10) becomes more time consuming for larger matrices. For example, it can require up to n^3 multiplications for each power of $\mathbf{A}$ needed beyond the first. This means that if $\mathbf{A}$ is annihalated by a polynomial of degree m then formula (3.10) can require at least $(m-2)n^3$ multiplications. In contrast, formula (4.11) typically requires at least $(n+1)n^2$ multiplications. This efficiency can be understood from the fact that if $\mathbf{A}$ is diagonalizable with $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ then

$$
\mathbf{A}^k = \mathbf{V}\mathbf{D}^k\mathbf{V}^{-1} \quad \text{for every positive integer } k .
$$

Once $\mathbf{A}$ has been diagonalized, we can compute $\mathbf{A}^k$ by first computing $\mathbf{D}^k$, which requires $(k-1)n$ multiplications, and then computing $\mathbf{V}\mathbf{D}^k\mathbf{V}^{-1}$, which requires another $(n+1)n^2$ multiplications. This contrasts with the $(k-1)n^3$ multiplications it takes to compute $\mathbf{A}^k$ directly.

4.6. **Generalized Eigenpairs.** So far the only method we have studied for computing a fundamental matrix of system (4.3) for a general $n\times n$ matrix $\mathbf{A}$ is the method for computing $e^{t\mathbf{A}}$ given in Section 3. The eigen methods presented so far in this section only yield n linearly independent solutions when $\mathbf{A}$ is either diagonalizable or 2×2. We now show how to extend eigen methods so that they yield n linearly independent solutions of system (4.3) for any $n\times n$ matrix $\mathbf{A}$.

Definition. If λ is an eigenvalue of a matrix $\mathbf{A}$ then a nonzero vector $\mathbf{v}$ that satisfies

$$(\mathbf{A}-\lambda\mathbf{I})^k\mathbf{v}=\mathbf{0} \quad \text{for some positive integer } k \tag{4.13}$$

is called a *generalized eigenvector* of $\mathbf{A}$ associated with λ, and $(\lambda,\mathbf{v})$ is called a *generalized eigenpair* of $\mathbf{A}$. The smallest k such that (4.13) holds is called its *degree*.

Remark. A generalized eigenvector has degree 1 if and only if it is an eigenvector.

The following fact generalizes recipe (4.6), which we used to construct a second solution for 2×2 matrices that are not diagonalizable.

Fact 10. If $(\lambda,\mathbf{v})$ is a generalized eigenpair of $\mathbf{A}$ of degree k then a solution of system (4.3) is

$$\mathbf{x}(t)=e^{\lambda t}\left(\mathbf{v}+t\,(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}+\cdots+\frac{t^{k-1}}{(k-1)!}\,(\mathbf{A}-\lambda\mathbf{I})^{k-1}\mathbf{v}\right). \tag{4.14}$$

Reason. By (4.13) and (4.14) we see that

$$\begin{aligned}(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}(t)&=e^{\lambda t}\left((\mathbf{A}-\lambda\mathbf{I})\mathbf{v}+t\,(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}+\cdots+\frac{t^{k-1}}{(k-1)!}\,(\mathbf{A}-\lambda\mathbf{I})^{k}\mathbf{v}\right)\\&=e^{\lambda t}\left((\mathbf{A}-\lambda\mathbf{I})\mathbf{v}+t\,(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}+\cdots+\frac{t^{k-2}}{(k-2)!}\,(\mathbf{A}-\lambda\mathbf{I})^{k-1}\mathbf{v}\right).\end{aligned}$$

Then differentiating (4.14) and using the above relation yields

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t)&=\lambda e^{\lambda t}\left(\mathbf{v}+t\,(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}+\cdots+\frac{t^{k-1}}{(k-1)!}\,(\mathbf{A}-\lambda\mathbf{I})^{k-1}\mathbf{v}\right)\\&\quad+e^{\lambda t}\left((\mathbf{A}-\lambda\mathbf{I})\mathbf{v}+\cdots+\frac{t^{k-2}}{(k-2)!}\,(\mathbf{A}-\lambda\mathbf{I})^{k-2}\mathbf{v}\right)\\&=\lambda\mathbf{x}(t)+(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}(t)\\&=\mathbf{A}\mathbf{x}(t).\end{aligned}$$

Therefore $\mathbf{x}(t)$ given by (4.14) is a solution of system (4.3). □

The reason the above recipe yields n linearly independent solutions of system (4.3) is the following fact from linear algebra.

Fact 11. If $\mathbf{A}$ is an $n\times n$ matrix and λ is an eigenvalue of $\mathbf{A}$ that is a root of $p_{\mathbf{A}}(z)$ with multiplicity m then there exists m generalized eigenpairs, $(\lambda,\mathbf{v}_1)$, $(\lambda,\mathbf{v}_2)$, $\cdots$, $(\lambda,\mathbf{v}_m)$, such that the generalized vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_m$ are linearly independent.

Remark. The proof of this fact is far beyond the scope of this course. However, you do not need to know the proof of this fact to use it.

Fact 11 does not give us a simple way to compute the m generalized eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_m$. If n is not too large then we can compute m linearly independent solutions of the algebraic system

$$(\mathbf{A} - \lambda\mathbf{I})^m \mathbf{v} = \mathbf{0}\,.$$

However if m and n are both not small then computing the powers $(\mathbf{A} - \lambda\mathbf{I})^m$ might take some time. In general we can start by looking for all solutions of

$$(\mathbf{A} - \lambda\mathbf{I}) \mathbf{v} = \mathbf{0}\,.$$

If there are m linearly independent solutions of this system then we are done. (This will be the case when $\mathbf{A}$ is diagonalizable.) Otherwise we can look for all solutions of

$$(\mathbf{A} - \lambda\mathbf{I})^2 \mathbf{v} = \mathbf{0}\,.$$

There will be solutions of this system that are not solutions of the previous system. If there are m linearly independent solutions of this system then we are done. Otherwise we continue to the next power. We will always find solutions of the system associated with each successive power that are not solutions of the previous system until we have found m linearly independent generalized eigenvectors. This can happen for a power much less than m.

One case for which there is a relatively direct algorithm for computing m linearly independent generalized eigenvectors is when there is only one linearly independent eigenvector associated with λ. In that case we can generate m linearly independent generalized eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_m$ by finding any nonzero solution of the system of m systems

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}\,, \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1\,, \quad \cdots \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_m = \mathbf{v}_{m-1}\,. \tag{4.15}$$

This system does not have a unique solution because we can add any multiple of $\mathbf{v}_1$ to each $\mathbf{v}_k$ and still have a solution. However, we do not need the general solution of this system — any solution will do. A set of generalized eigenvectors generated in this way is called an *eigenchain* of *length* m. It can be checked from (4.15) that for every $k = 1$, $\cdots$, m these vectors satisfy

$$(\mathbf{A} - \lambda\mathbf{I})^j \mathbf{v}_k = \mathbf{v}_{k-j} \quad \text{when } j < k\,, \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})^k \mathbf{v}_k = \mathbf{0}\,. \tag{4.16}$$

In particular, each $\mathbf{v}_k$ is a generalized eigenvector of degree k.

Given an eigenchain of length m associated with λ, we can generate m linearly independent solutions of the system (4.3) by

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{\lambda t}\mathbf{v}_1\,, \\
\mathbf{x}_2(t) &= e^{\lambda t}\big(\mathbf{v}_2 + t\,\mathbf{v}_1\big)\,, \\
\mathbf{x}_3(t) &= e^{\lambda t}\big(\mathbf{v}_3 + t\,\mathbf{v}_2 + \tfrac{1}{2}t^2\mathbf{v}_1\big)\,, \\
&\;\;\vdots \\
\mathbf{x}_m(t) &= e^{\lambda t}\Big(\mathbf{v}_m + t\,\mathbf{v}_{m-1} + \cdots + \tfrac{1}{(m-1)!}t^{m-1}\mathbf{v}_1\Big)\,,
\end{aligned} \tag{4.17}$$

Here each $\mathbf{x}_k(t)$ is obtained by setting $\mathbf{v} = \mathbf{v}_k$ in recipe (4.14) and using relations (4.16).

5. Linear Planar Systems

We now consider homogeneous linear systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{5.1}$$

where the coefficient matrix $\mathbf{A}$ is real and constant. Such a system is called *planar* because any solution of it can be thought of as tracing out a curve $(x(t), y(t))$ in the xy-plane.

5.1. Phase-Plane Portraits.

Of course, we have seen that solutions to system (5.1) can be expressed analytically as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} x_I \\ y_I \end{pmatrix}, \qquad \text{where} \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_I \\ y_I \end{pmatrix}, \tag{5.2}$$

and $e^{t\mathbf{A}}$ is given by one of the following three formulas that depend upon the roots of the characteristic polynomial $p(z) = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A})$.

- If $p(z)$ has simple real roots $\mu \pm \nu$ with $\nu \neq 0$ then

$$e^{t\mathbf{A}} = e^{\mu t}\left[\cosh(\nu t)\mathbf{I} + \frac{\sinh(\nu t)}{\nu}(\mathbf{A} - \mu\mathbf{I})\right]. \tag{5.3a}$$

- If $p(z)$ has conjugate roots $\mu \pm i\nu$ with $\nu \neq 0$ then

$$e^{t\mathbf{A}} = e^{\mu t}\left[\cos(\nu t)\mathbf{I} + \frac{\sin(\nu t)}{\nu}(\mathbf{A} - \mu\mathbf{I})\right]. \tag{5.3b}$$

- If $p(z)$ has a double real root μ then

$$e^{t\mathbf{A}} = e^{\mu t}\left[\mathbf{I} + t\,(\mathbf{A} - \mu\mathbf{I})\right]. \tag{5.3c}$$

While these analytic formulas are useful, we can gain insight into all solutions of system (5.1) by sketching a graph called its *phase-plane portrait* (or simply *phase portrait*).

As we have already observed, any solution of (5.1) can be thought of as tracing out a curve $(x(t), y(t))$ in the xy-plane — the so-called *phase-plane*. Each such curve is called an *orbit* or *trajectory* of the system.

Fact 1. Every point in the phase-plane has exactly one orbit passing through it.

Reason. This is a consequence of the existence and uniqueness theorem. □

Fact 1 implies that orbits fill the phase-plane, and that orbits cannot cross or otherwise share a point. We can gain insight into all solutions of system (5.1) by visualizing how their orbits fill the phase-plane.

Of course, the origin will be an orbit of system (5.1) for every $\mathbf{A}$. The solution that starts at the origin will stay at the origin. Points that give rise to solutions that do not move are called *stationary points*. In that case the entire orbit is a single point.

Remark. Stationary points go by many other names. They are sometimes called *equilibrium points*, *critical points*, or *fixed points*. This wealth of names is one measure of how important they are.

If the matrix $\mathbf{A}$ has a real eigenpair $(\lambda, \mathbf{v})$ then system (5.1) has special real solutions of the form

$$\mathbf{x}(t) = ce^{\lambda t}\mathbf{v}\,, \tag{5.4}$$

where c is any nonzero real constant. These solutions all lie on the line $\mathbf{x} = c\mathbf{v}$ parametrized by c. This line is easy to plot; it is simply the line that passes through the origin and the point $\mathbf{v}$. There are three possibilities.

- If $\lambda > 0$ then the line $\mathbf{x} = c\mathbf{v}$ consists of three orbits: the origin, corresponding to $c = 0$, plus the two remaining half-lines, corresponding to $c > 0$ and $c < 0$. Because $\lambda > 0$ all solutions on the half-lines will move away from the origin as t increases. We indicate this case by placing arrowheads that point away from the origin on each half-line pointing away. If $\mathbf{v}$ lies on the x-axis then it should look something like the following picture.

  If $\mathbf{v}$ does not lie on the x-axis then this picture should be rotated accordingly.

- If $\lambda < 0$ then the line $\mathbf{x} = c\mathbf{v}$ consists of three orbits: the origin, corresponding to $c = 0$, plus the two remaining half-lines, corresponding to $c > 0$ and $c < 0$. Because $\lambda < 0$ all solutions on the half-lines will move towards the origin as t increases. We indicate this case by placing arrowheads that point towards the origin on each half-line. If $\mathbf{v}$ lies on the x-axis then it should look something like the following picture.

  If $\mathbf{v}$ does not lie on the x-axis then this picture should be rotated accordingly.

- If $\lambda = 0$ then every point on the line $\mathbf{x} = c\mathbf{v}$ is a stationary point, and thereby is an orbit. We indicate this case by placing circles on each half-line. If $\mathbf{v}$ lies on the x-axis then it should look something like the following picture.

  If $\mathbf{v}$ does not lie on the x-axis then this picture should be rotated accordingly.

For every real eigenpair of $\mathbf{A}$ we should indicate these orbits in our phase-plane portrait before adding anything else. If $\mathbf{A}$ has no real eigenpairs then we are spared this step.

Remark. Because these solutions lie on the line $\mathbf{x} = c\mathbf{v}$ parametrized by c, they are sometimes called *line solutions*. This name can be a bit misleading because none of the solutions that lie on the line moves along the entire line.

Remark. After we have plotted the orbits corresponding to all the special solutions (5.4) we must complete the phase portrait by indicating what all the other orbits look like. In the next subsection we will do this using the analytic solutions (5.3a – 5.3c). Another approach to understanding the phase portrait is by plotting a direction field. This approach is illustrated in the textbook.

5.2. **Classification of Phase-Plane Portraits.** We now classify the twenty types of phase-plane portrait that can arise from a system of the form (5.1). The first step in this classification is determined by eigenvalues of the coefficient matrix $\mathbf{A}$. There are three cases: $\mathbf{A}$ has two real eigenvalues, $\mathbf{A}$ has a conjugate pair of eigenvalues, $\mathbf{A}$ has one real eigenvalue.

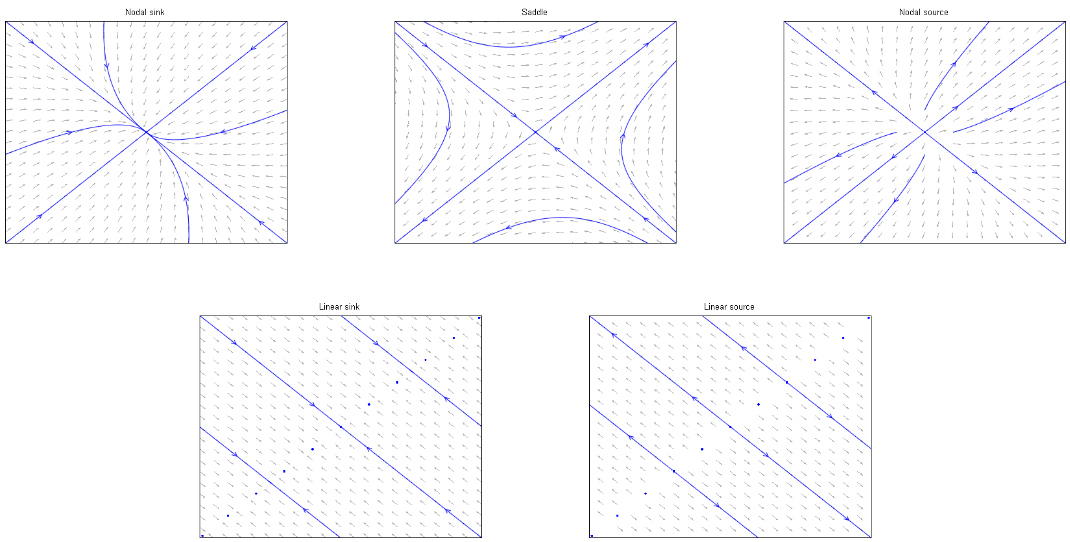
5.2.1. *Two Real Eigenvalues.* Suppose $\mathbf{A}$ has two real eigenvalues $\lambda_1 < \lambda_2$. Let $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ be real eigenpairs of $\mathbf{A}$. We first plot the ordits that lie on the lines $\mathbf{x} = c_1\mathbf{v}_1$ and $\mathbf{x} = c_2\mathbf{v}_2$ as described above. Every other solution of system (5.1) has the form

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \,, \tag{5.5}$$

where both c_1 and c_2 are arbitrary nonzero real numbers. There are five possible types of portrait.

- If $\lambda_1 < \lambda_2 < 0$ then every solution will approach the origin as $t \to \infty$. Because $e^{\lambda_1 t}$ decays to zero faster than $e^{\lambda_2 t}$ it is clear that the solution (5.5) behaves like $c_2 e^{\lambda_2 t}\mathbf{v}_2$ as $t \to \infty$. This means that all solutions not on the line $\mathbf{x} = c_1\mathbf{v}_1$ will approach the origin tangent to the line $\mathbf{x} = c_2\mathbf{v}_2$. This portrait is called a *nodal sink*.
- If $\lambda_1 < \lambda_2 = 0$ then the line $\mathbf{x} = c_2\mathbf{v}_2$ is a line of stationary points. It is clear that as $t \to \infty$ every solution (5.5) will approach one of these stationary points as along a line that is parallel to the line $\mathbf{x} = c_1\mathbf{v}_1$. This means that all solutions not on the line of stationary points $\mathbf{x} = c_2\mathbf{v}_2$ will approach that line along a line that is parallel to the line $\mathbf{x} = c_1\mathbf{v}_1$. This portrait is called a *linear sink*.
- If $\lambda_1 < 0 < \lambda_2$ then the nonzero solutions on the line $\mathbf{x} = c_1\mathbf{v}_2$ will approach the origin as $t \to \infty$ while the nonzero orbits on the line $\mathbf{x} = c_2\mathbf{v}_2$ will move away from the origin as t increases. It is clear that as $t \to \infty$ the solution (5.5) will approach the line $\mathbf{x} = c_2\mathbf{v}_2$ while as $t \to -\infty$ it will approach the line $\mathbf{x} = c_1\mathbf{v}_1$. This portrait is called a *saddle*.
- If $0 = \lambda_1 < \lambda_2$ then the line $\mathbf{x} = c_1\mathbf{v}_1$ is a line of stationary points. It is clear that as t increases the solution (5.5) will move away from one of these stationary points along a line that is parallel to the line $\mathbf{x} = c_2\mathbf{v}_2$. This means that all solutions not on the line of stationary points $\mathbf{x} = c_1\mathbf{v}_1$ will move away from that line along a line that is parallel to the line $\mathbf{x} = c_2\mathbf{v}_2$. This portrait is called a *linear source*.
- If $0 < \lambda_1 < \lambda_2$ then every solution will move away from the origin t increases. Because $e^{\lambda_2 t}$ decays to zero faster than $e^{\lambda_1 t}$ as $t \to -\infty$, it is clear that the solution (5.5) behaves like $c_1 e^{\lambda_1 t}\mathbf{v}_1$ as $t \to -\infty$. This means that all solutions not on the line $\mathbf{x} = c_2\mathbf{v}_2$ will emerge from the origin tangent to the line $\mathbf{x} = c_1\mathbf{v}_1$. This portrait is called a *nodal source*.

Examples. These five phase-plane portraits might look like the following.



In these five portraits the eigenvectors lie along the lines $y = x$ and $y = -x$. Nodal sinks, saddles, and nodal sources occur more often than linear sinks and linear sources.

5.2.2. *A Conjugate Pair of Eigenvalues.* Suppose $\mathbf{A}$ has a conjugate pair of eigenvalues $\mu \pm i\nu$ with $\nu \neq 0$. The analytic solution is

$$(5.6) \qquad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x_I \\ y_I \end{pmatrix} = e^{\mu t} \left[\mathbf{I} \cos(\nu t) + (\mathbf{A} - \mu \mathbf{I}) \frac{\sin(\nu t)}{\nu} \right] \begin{pmatrix} x_I \\ y_I \end{pmatrix} .$$

The matrix inside the square brackets is a periodic function of t with period $2\pi/\nu$.

When $\mu = 0$ this solution will trace out an ellipse. But will it do so with a clockwise or counterclockwise rotation? We can determine the direction of rotation by considering what happens at special values of $\mathbf{x}$.

$$\text{At } \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ we have } \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} .$$

This vector points up if $a_{21} > 0$, which indicates counterclockwise rotation. Similarly, this vector points down if $a_{21} < 0$, which indicates clockwise rotation.

$$\text{At } \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ we have } \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} .$$

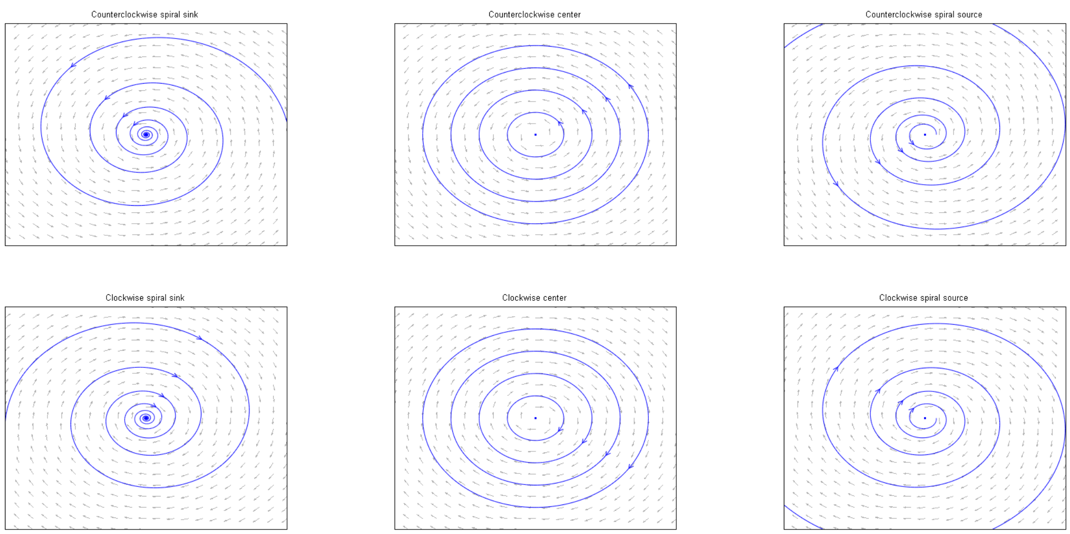
This vector points right if $a_{12} > 0$, which indicates clockwise rotation. Similarly, this vector points left if $a_{12} < 0$, which indicates counterclockwise rotation. Therefore we can read off the direction of rotation from the sign of either a_{21} or a_{12}.

We will use the sign of a_{21} to determine the direction of rotation because it is consistant with the sign convention of the *right-hand rule* from vector calculus. If $a_{21} > 0$ then point the thumb of your right hand up and the other fingers will indicated a counterclockwise rotation. If $a_{21} < 0$ then point the thumb of your right hand down and the other fingers will indicated a clockwise rotation.

It is clear from (5.6) that when $\mu < 0$ all solutions will approach the origin as $t \to \infty$, while when $\mu > 0$ all solutions will run away from the origin as t increases. If we put this together with the information above, we see there are six possible types of portrait.

- If $\mu < 0$ then all solutions spiral into the origin as $t \to \infty$. This portrait is called a *spiral sink*. The spiral is *counterclockwise* if $a_{21} > 0$, and is *clockwise* if $a_{21} < 0$.
- If $\mu = 0$ then all orbits are ellipses around the origin. This portrait is called a *center*. The center is *counterclockwise* if $a_{21} > 0$, and is *clockwise* if $a_{21} < 0$.
- If $\mu > 0$ then all solutions spiral away from the origin as $t \to \infty$. This portrait is called a *spiral source*. The spiral is *counterclockwise* if $a_{21} > 0$, and is *clockwise* if $a_{21} < 0$.

Examples. These six phase-plane portraits might look like the following



The left-hand portraits arise when $\operatorname{tr}(\mathbf{A}) < 0$. The center ones arise when $\operatorname{tr}(\mathbf{A}) = 0$. The right-hand ones arise when $\operatorname{tr}(\mathbf{A}) > 0$.

5.2.3. *One Real Eigenvalue.* Suppose $\mathbf{A}$ has one real eigenvalue μ. By (5.2) and (5.3c) the analytic solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x_I \\ y_I \end{pmatrix} = e^{\mu t} \left[\mathbf{I} + t\left(\mathbf{A} - \mu \mathbf{I}\right)\right] \begin{pmatrix} x_I \\ y_I \end{pmatrix} . \tag{5.7}$$

There are two subcases.

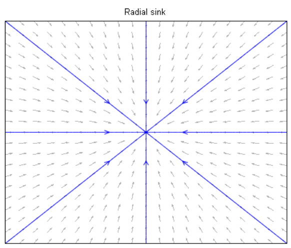
The simplest subcase is when $\mathbf{A} = \mu\mathbf{I}$. This subcase is easy to spot because the matrix $\mathbf{A}$ is simply a multipule of the identity matrix $\mathbf{I}$. It follows that every nonzero vector is an eigenvector of $\mathbf{A}$. In this subcase the analytic solution (5.7) reduces to

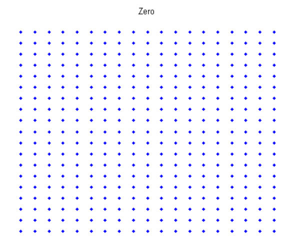
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x_I \\ y_I \end{pmatrix} = e^{\mu t} \begin{pmatrix} x_I \\ y_I \end{pmatrix} .$$

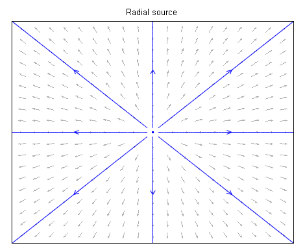
There are three possible types of portrait.

- If $\mu < 0$ all soutions radially approach the origin as $t \to \infty$. This portrait is called a *radial sink*.
- If $\mu = 0$ then all solutions are stationary. This portrait is called *zero*.
- If $\mu > 0$ all solutions radially move away from the origin as $t \to \infty$. This portrait is called a *radial source*.

Examples. These three phase-plane portraits might look like the following.





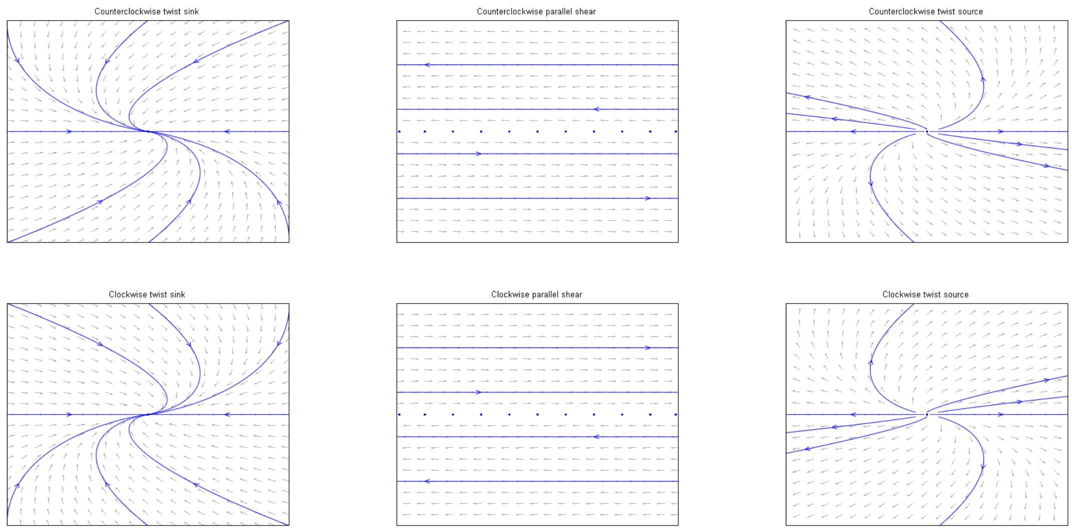


Because the subcase $\mathbf{A} = \mu\mathbf{I}$ is so special, these portraits do not arise as often as those for the next subcase.

The other subcase is when $\mathbf{A} \neq \mu\mathbf{I}$. In this subcase $\mathbf{A}$ will have the eigenpair $(\mu, \mathbf{v})$ where $\mathbf{v}$ is proportional to any nonzero column of $\mathbf{A} - \mu\mathbf{I}$. We first plot the ordits that lie on the line $\mathbf{x} = c\mathbf{v}$ as described above. There are six possible types of portrait.

- If $\mu < 0$ then all solutions on the line $\mathbf{x} = c\mathbf{v}$ move towards the origin as t increases. Because $e^{\mu t}$ will decay to zero faster than $e^{\mu t}t$, and because the columns of $\mathbf{A} - \mu\mathbf{I}$ are proportional to $\mathbf{v}$, it is clear from (5.7) that every solution approaches the origin tangent to the line $\mathbf{x} = c\mathbf{v}$. This portrait is called a *twist sink*. The twist is *counterclockwise* if either $a_{21} > 0$ or $a_{12} < 0$, and is *clockwise* if either $a_{21} < 0$ or $a_{12} > 0$.
- If $\mu = 0$ then all solutions on the line $\mathbf{x} = c\mathbf{v}$ are stationary. All other solutions will move parallel to this line. This portrait is called a *parallel shear*. The shear is *counterclockwise* if either $a_{21} > 0$ or $a_{12} < 0$, and is *clockwise* if either $a_{21} < 0$ or $a_{12} > 0$.
- If $\mu > 0$ then all solutions on the line $\mathbf{x} = c\mathbf{v}$ move away from the origin as t increases. Because $e^{\mu t}$ will decay to zero faster than $e^{\mu t}t$ as $t \to -\infty$, and because the columns of $\mathbf{A} - \mu\mathbf{I}$ are proportional to $\mathbf{v}$, it is clear from (5.7) that every solution emerges from the origin tangent to the line $\mathbf{x} = c\mathbf{v}$. This portrait is called a *twist source*. The twist is *counterclockwise* if either $a_{21} > 0$ or $a_{12} < 0$, and is *clockwise* if either $a_{21} < 0$ or $a_{12} > 0$.

Examples. These six phase-plane portraits might look like the following.



In these six portraits the eigenvectors lie along the line $y = 0$. Twist sinks and twist sources occur more often than parallel shears.

Remark. For a twist or a shear we can have $a_{21} = 0$ or $a_{12} = 0$ but we cannot have $a_{21} = a_{12} = 0$. Therefore we will always be able to determine the direction of rotation from either a_{21} or a_{12}. Alternatively, we can combine the a_{21} test and the a_{12} test into a single test on $a_{21} - a_{12}$ that works for every spiral, center, twist, or shear. Namely, if $a_{21} - a_{12} > 0$ the rotation is counterclockwise, while if $a_{21} - a_{12} < 0$ the rotation is clockwise. This can be recalled with that aid of the "right-hand" rule.

Remark. The textbook calls radial sinks and sources *proper nodes* and twist sinks and sources *improper nodes*. While traditional, this terminology is both more cumbersome and much less desciptive than that used here. Moreover, it often leaves students with the impression that improper nodes are rare. However, the opposite is true; improper nodes are much more common than proper nodes. Other books refer to radial sinks and sources as *star* sinks and sources.

Remark. The textbook fails to include linear sinks and sources, parallel shears, or zero in its classification of types of phase portraits. These are the types for which $\det(\mathbf{A}) = 0$. There is no good justification for excluding these types.

5.3. **Stability of the Origin.** The origin of the phase-plane plays a special role for linear systems. This is because any solution $(x(t), y(t))$ of system (5.1) that starts at the origin will stay at the origin. In other words, the origin is an orbit for every linear system (5.1).

> **Definition 5.1.** We say that the origin is *stable* if every solution that starts sufficiently near it will stay arbitrarily close to it. We say that the origin is *unstable* if it is not stable.

The language "every solution that starts sufficiently near it will stay arbitrarily close to it" is not very precise. Rather than formulate a more precise mathematical definition, we will build our understanding of these notions through examples. To begin with, it should be clear that for every system the origin is either stable or unstable. Roughly speaking, the origin will be unstable if at least one solution that starts near it will move away from it.

Definition 5.2. We say that the origin is *attracting* if every solution that starts near it will approach it as $t \to \infty$. We say that the origin is *repelling* if every solution that starts near it but not at it will move away from it.

It should be clear that if the origin is attracting then it is stable, and that if it is repelling then it is unstable. These implications do not go the other way. Indeed, we will soon see that there are systems for which the origin is stable but not attracting, and systems which the origin is unstable but not repelling.

When we examine the different types of phase-plane portrait we see that the stability of the origin for each of them is as given by the following table.

attracting (so also stable)	stable (but not attracting)	unstable (but not repelling)	repelling (so also unstable)
nodal sinks	centers	saddles	nodal sources
radial sinks	linear sinks	linear sources	radial sources
twist sinks	zero	parallel shears	twist sources
spiral sinks			spiral sources

The book calls attracting *asymptotically stable* but has no terminology for repelling.

5.4. **Mean-Discriminant Plane.** We can visualize the relationships between various types of phase-plane portrait for linear systems through the mean-discriminant plane. By completing the square of the characteristic polynomial we bring it into the form

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = (z-\mu)^2 - \delta\,,$$

where the *mean* μ and *discriminant* δ are given by

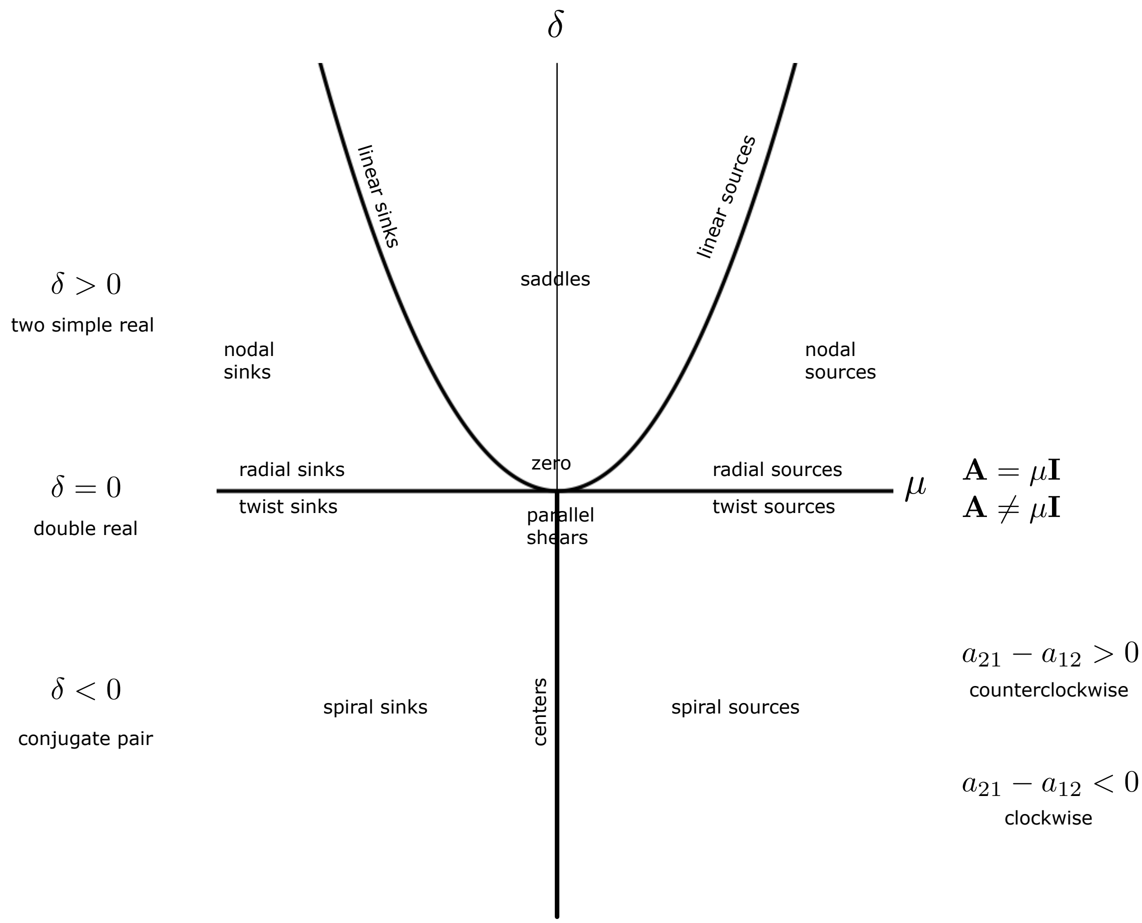
$$\mu = \tfrac{1}{2}\operatorname{tr}(\mathbf{A})\,, \qquad \delta = \mu^2 - \det(\mathbf{A})\,.$$

Recall that δ is called the discriminant because it determines the root structure of the characteristic polynomial:

- when $\delta > 0$ there are the two simple real roots $\mu \pm \sqrt{\delta}$;
- when $\delta = 0$ there is the one double real root μ;
- when $\delta < 0$ there is the conjugate pair of roots $\mu \pm i\sqrt{|\delta|}$.

In all cases μ is the average of the roots, which is why it is called the mean.

The types of phase-plane portrait that arise for different values of μ and δ in the $\mu\delta$-plane are shown in the figure below.



In the upper half-plane ($\delta > 0$) the matrix $\mathbf{A}$ has the two simple real eigenvalues $\mu \pm \sqrt{\delta}$. These will both be negative whenever $\mu < -\sqrt{\delta}$, which is the region labeled *nodal sinks* in the figure. These will both be positive whenever $\sqrt{\delta} < \mu$, which is the region labeled *nodal sources* in the figure. These will have opposite signs whenever $-\sqrt{\delta} < \mu < \sqrt{\delta}$, which is the region labeled *saddles* in the figure. These three regions are seperated by the parabola $\delta = \mu^2$. There is one negative and one zero eigenvalue along the branch of this parabola where $\mu = -\sqrt{\delta}$, which is labeled *linear sinks* in the figure. There is one zero and one positive eigenvalue along the branch of this parabola where $\mu = \sqrt{\delta}$, which is labeled *linear sources* in the figure. There are five types of phase portrait in the upper half-plane

In the lower half-plane ($\delta < 0$) the matrix $\mathbf{A}$ has the conjugate pair of eigenvalues $\mu \pm i\sqrt{|\delta|}$. These will have negative real parts whenever $\mu < 0$, which is the region labeled *spiral sinks* in the figure. These will have positive real parts whenever $\mu > 0$, which is the region labeled *spiral sources* in the figure. These will be purely imaginary whenever $\mu = 0$, which is the half-line labeled *centers* in the figure. In each case we can determine the rotation of the orbits (*counterclockwise* or *clockwise*) by the a_{21} test. There are six types of phase portrait in the lower half-plane

On the μ-axis ($\delta = 0$) the matrix $\mathbf{A}$ has the single real eigenvalue μ. There are two cases to consider: the case when $\mathbf{A} = \mu\mathbf{I}$ which we label above the μ-axis, and the case when $\mathbf{A} \neq \mu\mathbf{I}$ which we label below the μ-axis.

- When $\mathbf{A} = \mu\mathbf{I}$ every nonzero vector is an eigenvector associated with the eigenvalue μ. This eigenvalue is negative whenever $\mu < 0$, which is the half-line labeled *radial sink* in the figure. This eigenvalue is positive whenever $\mu > 0$, which is the half-line labeled *radial source* in the figure. This eigenvalue is zero whenever $\mu = 0$, in which case $\mathbf{A} = \mu\mathbf{I} = 0$, which is why the origin is labeled *zero* in the figure. There are three types of phase portrait represented here.
- When $\mathbf{A} \neq \mu\mathbf{I}$ the eigenvectors associated with the eigenvalue μ are proportional to any nonzero column of $\mathbf{A} - \mu\mathbf{I}$. This eigenvalue is negative whenever $\mu < 0$, which is the half-line labeled *twist sink* in the figure. This eigenvalue is positive whenever $\mu > 0$, which is the half-line labeled *twist source* in the figure. This eigenvalue is zero whenever $\mu = 0$, in which case the origin is labeled *parallel shears* in the figure. In each of these subcases we can determine the rotation of the orbits (*counterclockwise* or *clockwise*) either by the a_{21} test or the a_{12} test or by the $a_{21} - a_{12}$ test. There six types of phase portrait represented here.

Remark. The textbook and many other books present a similar picture, but use the trace-determinant plane rather than the mean-discriminant plane. Therefore the pictures are not the same! The advantage of the mean-discriminant plane picture is its clean separation of the cases of two real, one real, and conjugate pair eigenvalues into the upper half-plane, μ-axis, and lower half-plane respectively. This allows a cleaner presentation of the tests for the rotation of the orbits (counterclockwise or clockwise). It also allows a cleaner presentation of the nine types of phase portrait on the μ-axis.