

**FIRST-ORDER SYSTEMS OF  
ORDINARY DIFFERENTIAL EQUATIONS I:  
Introduction and Linear Systems**

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Because the presentation of this material in lecture will differ from that in the book, I felt that notes that closely follow the lecture presentation might be appreciated.

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## 1. INTRODUCTION TO FIRST-ORDER SYSTEMS

**1.1. Normal Forms and Solutions.** For the remainder of the course we will study first-order systems of  $n$  ordinary differential equations for functions  $x_j(t)$ ,  $j = 1, 2, \dots, n$  that can be put into the normal form

$$(1.1) \quad \begin{aligned} \frac{dx_1}{dt} &= f_1(t, x_1, x_2, \dots, x_n), \\ \frac{dx_2}{dt} &= f_2(t, x_1, x_2, \dots, x_n), \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(t, x_1, x_2, \dots, x_n). \end{aligned}$$

We say that  $n$  is the *dimension* of this system.

System (1.1) can be expressed more compactly in *vector notation* as

$$(1.2) \quad \frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),$$

where  $\mathbf{x}$  and  $\mathbf{f}(t, \mathbf{x})$  are given by the  $n$ -dimensional *column vectors*

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, x_1, x_2, \dots, x_n) \\ f_2(t, x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(t, x_1, x_2, \dots, x_n) \end{pmatrix}.$$

We thereby express the system of  $n$  equations (1.1) as the single vector equation (1.2). We say  $x_1, x_2, \dots, x_n$  are the *entries* of the vector  $\mathbf{x}$ . Similarly, we say that the functions  $f_1(t, x_1, x_2, \dots, x_n), f_2(t, x_1, x_2, \dots, x_n), \dots, f_n(t, x_1, x_2, \dots, x_n)$  are the entries of the vector-valued function  $\mathbf{f}(t, \mathbf{x})$ .

**Remark.** We will use boldface, lowercase letters like  $\mathbf{x}$  and  $\mathbf{f}$  to denote column vectors. Other common notations include an underline like  $\underline{x}$  and  $\underline{f}$ , or an arrow like  $\vec{x}$  and  $\vec{f}$ . Some advanced books do not use any special notation for vectors, but expect the reader to recall what each letter represents from when it was introduced.

You should recall from multi-variable Calculus what it means for a vector-valued function  $\mathbf{u}(t)$  to be either continuous or differentiable at a point.

- We say  $\mathbf{u}(t)$  is *continuous* at time  $t$  if *every entry* of  $\mathbf{u}(t)$  is continuous at  $t$ .
- We say  $\mathbf{u}(t)$  is *differentiable* at time  $t$  if *every entry* of  $\mathbf{u}(t)$  is differentiable at  $t$ .

Given these definitions, we define what it means for a vector-valued function  $\mathbf{u}(t)$  to be either continuous, differentiable, or continuously differentiable over a time interval.

- We say  $\mathbf{u}(t)$  is *continuous over* a time interval  $(t_L, t_R)$  if it is continuous at every  $t$  in  $(t_L, t_R)$ .
- We say  $\mathbf{u}(t)$  is *differentiable over* a time interval  $(t_L, t_R)$  if it is differentiable at every  $t$  in  $(t_L, t_R)$ .
- We say  $\mathbf{u}(t)$  is *continuously differentiable over* a time interval  $(t_L, t_R)$  if it is differentiable over  $(t_L, t_R)$  and its derivative is continuous over  $(t_L, t_R)$ .

We are now ready to define what we mean by a solution of system (1.2).

**Definition.** We say that  $\mathbf{x}(t)$  is a *solution* of system (1.2) over a time interval  $(t_L, t_R)$  when

1.  $\mathbf{x}(t)$  is differentiable at every  $t$  in  $(t_L, t_R)$ ;
2.  $\mathbf{f}(t, \mathbf{x}(t))$  is defined for every  $t$  in  $(t_L, t_R)$ ;
3. equation (1.2) holds at every  $t$  in  $(t_L, t_R)$ .

**Remark.** This definition is similar to definitions of solutions to single differential equations that we gave earlier. The first point states that the right-hand side of the equation makes sense. The second point states that the left-hand side of the equation makes sense. The third point states that the two sides are equal.

**1.2. Initial-Value Problems.** We will consider initial-value problems of the form

$$(1.3) \quad \frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_I) = \mathbf{x}^I.$$

Here  $t_I$  is the *initial time*,  $\mathbf{x}^I$  is the *initial value* or *initial data*, and  $\mathbf{x}(t_I) = \mathbf{x}^I$  is the *initial condition*. Below we will give conditions on  $\mathbf{f}(t, \mathbf{x})$  that insure this problem has a unique solution that exists over some time interval that contains  $t_I$ . We begin with a definition.

**Definition 1.1.** Let  $S$  be a set in  $\mathbb{R} \times \mathbb{R}^n$ . A point  $(t_o, \mathbf{x}_o)$  is said to be in the interior of  $S$  if there exists a box  $(t_L, t_R) \times (x_1^L, x_1^R) \times \cdots \times (x_n^L, x_n^R)$  that contains the point  $(t_o, \mathbf{x}_o)$  and also lies within the set  $S$ .

Our basic existence and uniqueness theorem is the following.

**Theorem 1.1.** Let  $\mathbf{f}(t, \mathbf{x})$  be a vector-valued function defined over a set  $S$  in  $\mathbb{R} \times \mathbb{R}^n$  such that

- $\mathbf{f}$  is continuous over  $S$ ,
- $\mathbf{f}$  is differentiable with respect to each  $x_i$  over  $S$ ,
- each  $\partial_{x_i} \mathbf{f}$  is continuous over  $S$ .

Then for every initial time  $t_I$  and every initial value  $\mathbf{x}^I$  such that  $(t_I, \mathbf{x}^I)$  is in the interior of  $S$  there exists a unique solution  $\mathbf{x}(t)$  to initial-value problem (1.3) that is defined over some time interval  $(a, b)$  such that

- $t_I$  is in  $(a, b)$ ,
- $\{(t, \mathbf{x}(t)) : t \in (a, b)\}$  lies within the interior of  $S$ .

Moreover,  $\mathbf{x}(t)$  extends to the largest such time interval and  $\mathbf{x}'(t)$  is continuous over that time interval.

**Remark.** This is not the most general theorem we could state, but it applies easily to the first-order you will face in this course. It asserts that the initial-value problem (1.3) has a unique solution  $\mathbf{x}(t)$  that will exist until  $(t, \mathbf{x}(t))$  leaves the interior of  $S$ .

**1.3. Recasting Higher-Order Problems as First-Order Systems.** Many higher-order differential equation problems can be recast in terms of a first-order system in the normal form (1.2). For example, every  $n^{\text{th}}$ -order ordinary differential equation in the normal form

$$y^{(n)} = g(t, y, y', \dots, y^{(n-1)}) ,$$

can be expressed as an  $n$ -dimensional first-order system in the form (1.2) with

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ g(t, x_1, x_2, \dots, x_n) \end{pmatrix}, \quad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} .$$

Notice that the first-order system is expressed solely in terms of the entries of  $\mathbf{x}$ . The “dictionary” that relates  $\mathbf{x}$  to  $y, y', \dots, y^{(n-1)}$  is given as a separate equation.

**Example.** Recast as a first-order system

$$y''' + yy' + e^t y^2 = \cos(3t) .$$

**Solution.** Because this single equation is third order, the first-order system will have dimension three. It will be

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \cos(3t) - x_1 x_2 - e^t x_1^2 \end{pmatrix}, \quad \text{where } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} .$$

More generally, every  $d$ -dimensional  $m^{\text{th}}$ -order ordinary differential system in the normal form

$$\mathbf{y}^{(m)} = \mathbf{g}(t, \mathbf{y}, \mathbf{y}', \dots, \mathbf{y}^{(n-1)}) ,$$

can be expressed as an  $md$ -dimensional first-order system in the form (1.2) with

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \\ \mathbf{g}(t, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) \end{pmatrix}, \quad \text{where } \mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \end{pmatrix} = \begin{pmatrix} \mathbf{y} \\ \mathbf{y}' \\ \vdots \\ \mathbf{y}^{(m-1)} \end{pmatrix} .$$

Here each  $\mathbf{x}_k$  is a  $d$ -dimensional vector while  $\mathbf{x}$  is the  $md$ -dimensional vector constructed by stacking the vectors  $\mathbf{x}_1$  through  $\mathbf{x}_m$  on top of each other.

**Example.** Recast as a first-order system

$$q_1'' + f_1(q_1, q_2) = 0, \quad q_2'' + f_2(q_1, q_2) = 0 .$$

**Solution.** Because this two dimensional system is second order, the first-order system will have dimension four. It will be

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ -f_1(x_1, x_2) \\ -f_2(x_1, x_1) \end{pmatrix}, \quad \text{where } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_1' \\ q_2' \end{pmatrix} .$$

When faced with a higher-order initial-value problem, we use the dictionary to obtain the initial values for the first-order system from those for the higher-order problem.

**Example.** Recast as an initial-value problem for a first-order system

$$y'''' - e^y = 0, \quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = 5, \quad y'''(0) = -4.$$

**Solution.** The first-order initial-value problem is

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ e^{x_1} \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \\ -4 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix},$$

**Remark.** We can also find single higher-order equations that are satisfied by the entries of a first-order system. We will not discuss how this is done because it is not as useful.

**1.4. Numerical Methods.** One advantage of expressing an initial-value problem in the form of a first-order system is that we can then apply all the numerical methods that we studied earlier in the setting of single first-order equations. In fact, the most common way in which numerical methods are applied to construct a numerical approximation of the solution to an initial-value problem for a higher-order equation is to recast it as an initial-value problem for a first-order system and then apply such numerical methods.

Suppose we wish to construct a numerical approximation over the time interval  $[t_I, t_F]$  to the solution  $\mathbf{x}(t)$  of the initial-value problem

$$(1.4) \quad \frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_I) = \mathbf{x}^I.$$

A numerical method selects times  $\{t_n\}_{n=0}^N$  such that

$$t_I = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = t_F,$$

and computes vectors  $\{\mathbf{x}_n\}_{n=0}^N$  such that  $\mathbf{x}_0 = \mathbf{x}^I$  and  $\mathbf{x}_n$  approximates  $\mathbf{x}(t_n)$  for  $n = 1, 2, \dots, N$ . If we do this by using  $N$  uniform time steps (as we did earlier) then we set

$$h = \frac{t_F - t_I}{N}, \quad \text{and} \quad t_n = t_I + nh \quad \text{for } n = 0, 1, \dots, N,$$

where  $h$  is called the *time step*. Below we show that the vectors  $\{\mathbf{x}_n\}_{n=0}^N$  can be computed easily by the four explicit methods that we studied earlier for scalar-valued equations: the explicit Euler, Runge-trapezoidal, Runge-midpoint, and Runge-Kutta methods. The only modification that we need to make in these explicit methods is to replace the scalar-valued dependent variables and functions with vector-valued ones. The justifications of these methods also carry over upon making the same modification.

**Remark.** Implicit methods such as the implicit Euler method are often extremely useful for computing approximate solutions for first-order systems, but are more complicated to implement because they require the numerical solution of algebraic systems, which is beyond the scope of this course.

*Explicit Euler Method.* The vector-valued version of this method is as follows.

Set  $\mathbf{x}_0 = \mathbf{x}^I$  and then for  $n = 0, \dots, N - 1$  cycle through

$$\mathbf{f}_n = \mathbf{f}(t_n, \mathbf{x}_n), \quad \mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}_n,$$

where  $t_n = t_I + nh$ .

**Remark.** Like its scalar-valued version, this method is first-order.

*Runge-Trapezoidal Method.* The vector-valued version of this method is as follows.

Set  $\mathbf{x}_0 = \mathbf{x}^I$  and then for  $n = 0, \dots, N - 1$  cycle through

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n), & \tilde{\mathbf{x}}_{n+1} &= \mathbf{x}_n + h\mathbf{f}_n, \\ \tilde{\mathbf{f}}_{n+1} &= \mathbf{f}(t_{n+1}, \tilde{\mathbf{x}}_{n+1}), & \mathbf{x}_{n+1} &= \mathbf{x}_n + \frac{1}{2}h[\mathbf{f}_n + \tilde{\mathbf{f}}_{n+1}], \end{aligned}$$

where  $t_n = t_I + nh$ .

**Remark.** Like its scalar-valued version, this method is second-order. It requires twice as many function evaluations per time step as the explicit Euler method. Because it is second order, this method often outperforms the explicit Euler method because the same error often can be realized with a time step that is more than twice as large.

*Runge-Midpoint Method.* The vector-valued version of this method is as follows.

Set  $\mathbf{x}_0 = \mathbf{x}^I$  and then for  $n = 0, \dots, N - 1$  cycle through

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n), & \mathbf{x}_{n+\frac{1}{2}} &= \mathbf{x}_n + \frac{1}{2}h\mathbf{f}_n, \\ \mathbf{f}_{n+\frac{1}{2}} &= \mathbf{f}(t_{n+\frac{1}{2}}, \mathbf{x}_{n+\frac{1}{2}}), & \mathbf{x}_{n+1} &= \mathbf{x}_n + h\mathbf{f}_{n+\frac{1}{2}}, \end{aligned}$$

where  $t_n = t_I + nh$  and  $t_{n+\frac{1}{2}} = t_I + (n + \frac{1}{2})h$ .

**Remark.** Like its scalar-valued version, this method is second-order. It has the same number of function evaluations per time step as the Runge-trapezoidal method. Because they are the same order, their performances are comparable.

*Runge-Kutta Method.* The vector-valued version of this method is as follows.

Set  $\mathbf{x}_0 = \mathbf{x}^I$  and then for  $n = 0, \dots, N - 1$  cycle through

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n), & \tilde{\mathbf{x}}_{n+\frac{1}{2}} &= \mathbf{x}_n + \frac{1}{2}h\mathbf{f}_n, \\ \tilde{\mathbf{f}}_{n+\frac{1}{2}} &= \mathbf{f}(t_{n+\frac{1}{2}}, \tilde{\mathbf{x}}_{n+\frac{1}{2}}), & \mathbf{x}_{n+\frac{1}{2}} &= \mathbf{x}_n + \frac{1}{2}h\tilde{\mathbf{f}}_{n+\frac{1}{2}}, \\ \mathbf{f}_{n+\frac{1}{2}} &= \mathbf{f}(t_{n+\frac{1}{2}}, \mathbf{x}_{n+\frac{1}{2}}), & \tilde{\mathbf{x}}_{n+1} &= \mathbf{x}_n + h\mathbf{f}_{n+\frac{1}{2}}, \\ \tilde{\mathbf{f}}_{n+1} &= \mathbf{f}(t_{n+1}, \tilde{\mathbf{x}}_{n+1}), & \mathbf{x}_{n+1} &= \mathbf{x}_n + \frac{1}{6}h[\mathbf{f}_n + 2\tilde{\mathbf{f}}_{n+\frac{1}{2}} + 2\mathbf{f}_{n+\frac{1}{2}} + \tilde{\mathbf{f}}_{n+1}], \end{aligned}$$

where  $t_n = t_I + nh$  and  $t_{n+\frac{1}{2}} = t_I + (n + \frac{1}{2})h$ .

**Remark.** Like its scalar-valued version, this method is fourth-order. It requires twice as many function evaluations per time step as either second-order method and four times more than the explicit Euler method. However, because it is fourth order, the same error often can be realized with a time step that is more than twice as large as that for either second-order method. In such cases, this method outperforms all the foregoing methods. Variants of this method are among the most widely used numerical methods for approximating the solution of initial-value problems. The variant used by the MATLAB command “ode45” is the Dormand-Prince method.

**Remark.** In addition to the four methods given above, here are vector-valued versions of the classical third-order methods due to *Heun* and *Kutta*.

*Heun Method.* Set  $\mathbf{x}_0 = \mathbf{x}^I$  and then for  $n = 0, \dots, N - 1$  cycle through

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n), & \mathbf{x}_{n+\frac{1}{3}} &= \mathbf{x}_n + \frac{1}{3}h\mathbf{f}_n, \\ \mathbf{f}_{n+\frac{1}{3}} &= \mathbf{f}(t_{n+\frac{1}{3}}, \mathbf{x}_{n+\frac{1}{3}}), & \mathbf{x}_{n+\frac{2}{3}} &= \mathbf{x}_n + \frac{2}{3}h\mathbf{f}_{n+\frac{1}{3}}, \\ \mathbf{f}_{n+\frac{2}{3}} &= \mathbf{f}(t_{n+\frac{2}{3}}, \mathbf{x}_{n+\frac{2}{3}}), & \mathbf{x}_{n+1} &= \mathbf{x}_n + \frac{1}{4}h[\mathbf{f}_n + 3\mathbf{f}_{n+\frac{2}{3}}], \end{aligned}$$

where  $t_n = t_I + nh$ ,  $t_{n+\frac{1}{3}} = t_I + (n + \frac{1}{3})h$ , and  $t_{n+\frac{2}{3}} = t_I + (n + \frac{2}{3})h$ .

*Kutta Method.* Set  $\mathbf{x}_0 = \mathbf{x}^I$  and then for  $n = 0, \dots, N - 1$  cycle through

$$\begin{aligned} \mathbf{f}_n &= \mathbf{f}(t_n, \mathbf{x}_n), & \mathbf{x}_{n+\frac{1}{2}} &= \mathbf{x}_n + \frac{1}{2}h\mathbf{f}_n, \\ \mathbf{f}_{n+\frac{1}{2}} &= \mathbf{f}(t_{n+\frac{1}{2}}, \mathbf{x}_{n+\frac{1}{2}}), & \tilde{\mathbf{x}}_{n+1} &= \mathbf{x}_n + h[-\mathbf{f}_n + 2\mathbf{f}_{n+\frac{1}{2}}], \\ \tilde{\mathbf{f}}_{n+1} &= \mathbf{f}(t_{n+1}, \tilde{\mathbf{x}}_{n+1}), & \mathbf{x}_{n+1} &= \mathbf{x}_n + \frac{1}{6}h[\mathbf{f}_n + 4\mathbf{f}_{n+\frac{1}{2}} + \tilde{\mathbf{f}}_{n+1}], \end{aligned}$$

where  $t_n = t_I + nh$  and  $t_{n+\frac{1}{2}} = t_I + (n + \frac{1}{2})h$ .

**1.5. Application: Tank Problems.** First-order systems arise in many applications. In this section we show they arise from problems of describing interconnected tanks. These represent a broad class of problems that describe the transport of some quantity into and out of tanks or other volumes. The quantity might be a fluid like water, oil, or air, or it might be a substance like a solute or pollutant that is carried along by a fluid. The tanks might be well-defined volumes like ponds, lakes, or rooms in a building. These problems lie at the heart of many numerical simulations of fluids.

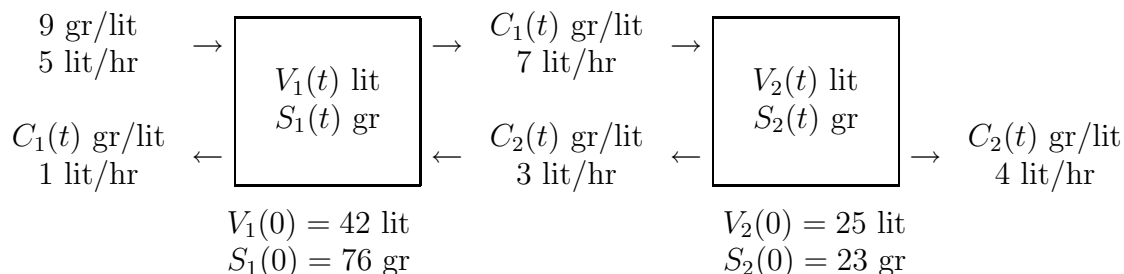
In such problems we construct an initial-value problem satisfied by the amounts  $Q_i$  of some quantity in the tank  $i$ . The associated system of ordinary differential equation will consist of equations in the form

$$\frac{dQ_i}{dt} = \text{RATE IN}_i - \text{RATE OUT}_i,$$

where  $\text{RATE IN}_i$  is the rate the quantity enters tank  $i$  while  $\text{RATE OUT}_i$  is the rate the quantity exits the tank  $i$ . There will be one such equation for each tank. For some problems  $\text{RATE IN}_i$  and  $\text{RATE OUT}_i$  will be given explicitly in the problem. At other times they will be given in terms of other variables in the problem. The way in which this is done is similar to the way we did it for the tank problems we studied earlier that involved just one tank.

**Example.** Consider two interconnected tanks filled with brine (salt water). The first tank contains 42 liters and the second contains 25 liters. Brine with a salt concentration of 9 grams per liter flows into the first tank at 5 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 3 liters per hour, from the first into a drain at 1 liter per hour, and from the second into a drain at 4 liters per hour. At  $t = 0$  there are 76 grams of salt in the first tank and 23 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

**Solution.** Let  $V_1(t)$  and  $V_2(t)$  be the volumes (lit) of brine in the first and second tank at time  $t$  minutes. Let  $S_1(t)$  and  $S_2(t)$  be the mass (gr) of salt in the first and second tank at time  $t$  minutes. Because mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time  $t$  are  $C_1(t) = S_1(t)/V_1(t)$  and  $C_2(t) = S_2(t)/V_2(t)$  respectively. In particular, these will be the concentrations of the brine that flows out of the respective tank. We have the following picture.



We are asked to write down an initial-value problem that governs  $S_1(t)$  and  $S_2(t)$ .

The rates work out so there will always be  $V_1(t) = 42$  liters of brine in the first tank and  $V_2(t) = 25$  liters in the second. Then  $S_1(t)$  and  $S_2(t)$  are governed by the initial-value problem

$$\begin{aligned} \frac{dS_1}{dt} &= 9 \cdot 5 + \frac{S_2}{25} 3 - \frac{S_1}{42} 7 - \frac{S_1}{42} 1, & S_1(0) &= 76, \\ \frac{dS_2}{dt} &= \frac{S_1}{42} 7 - \frac{S_2}{25} 3 - \frac{S_2}{25} 4, & S_2(0) &= 23. \end{aligned}$$

This can be simplified to

$$\begin{aligned} \frac{dS_1}{dt} &= 45 + \frac{3}{25} S_2 - \frac{4}{21} S_1, & S_1(0) &= 76, \\ \frac{dS_2}{dt} &= \frac{1}{6} S_1 - \frac{7}{25} S_2, & S_2(0) &= 23. \end{aligned}$$

**Remark.** The system of ordinary differential equations derived in the previous example is linear, which is the type of system we will study next.



## 2. LINEAR SYSTEMS: GENERAL METHODS AND THEORY

The  $n$ -dimensional first-order system (1.1) is called *linear* when it has the form

$$(2.1) \quad \begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t), \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t), \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t). \end{aligned}$$

The functions  $a_{jk}(t)$  are called *coefficients* while the functions  $f_j(t)$  are called *forcings*.

We can use *matrix notation* to compactly write linear system (2.1) as

$$(2.2) \quad \frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t),$$

where  $\mathbf{x}$  and  $\mathbf{f}(t)$  are the  $n$ -dimensional column vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix},$$

while  $\mathbf{A}(t)$  is the  $n \times n$  matrix

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}.$$

We call  $\mathbf{A}(t)$  the *coefficient matrix* and  $\mathbf{f}(t)$  the *forcing vector*. System (2.2) is said to be *homogeneous* if  $\mathbf{f}(t) = \mathbf{0}$  and *nonhomogeneous* otherwise.

The product  $\mathbf{A}(t)\mathbf{x}$  appearing in system (2.2) denotes column vector that results from the *matrix multiplication* of the matrix  $\mathbf{A}(t)$  with the column vector  $\mathbf{x}$ . The sum appearing in (2.2) denotes column vector that results from the *matrix addition* of the column vector  $\mathbf{A}(t)\mathbf{x}$  with the column vector  $\mathbf{f}(t)$ . These matrix operations are presented in Appendix B.

**2.1. Initial-Value Problems.** We will consider linear initial-value problems in the form

$$(2.3) \quad \frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(t_I) = \mathbf{x}^I,$$

where  $\mathbf{x}^I$  is called the *vector of initial values*, or simply the *initial vector*.

A major theme of this section is that for every fact that we studied about higher-order linear equations there is an analogous fact about linear first-order systems. For example, the basic existence and uniqueness theorem is the following.

**Theorem 2.1.** If  $\mathbf{A}(t)$  and  $\mathbf{f}(t)$  are continuous over the time interval  $(t_L, t_R)$  then for every initial time  $t_I$  in  $(t_L, t_R)$  and every initial vector  $\mathbf{x}^I$  the initial-value problem (2.3) has a unique solution  $\mathbf{x}(t)$  that is continuously differentiable over  $(t_L, t_R)$ . Moreover, if  $\mathbf{A}(t)$  and  $\mathbf{f}(t)$  are  $k$ -times continuously differentiable over the time interval  $(t_L, t_R)$  then  $\mathbf{x}(t)$  will be is  $(k + 1)$ -times continuously differentiable over  $(t_L, t_R)$ .

You should be able to use the Basic Existence and Uniqueness Theorem to identify the interval of definition for solutions of the initial-value problem (2.3). This is done very much like the way you identified intervals of definition for solutions of higher-order linear equations. Specifically, if  $\mathbf{x}(t)$  is the solution of the initial-value problem (2.3) then its interval of definition will be  $(t_L, t_R)$  whenever:

- every entry of the coefficient matrix  $\mathbf{A}(t)$  and the forcing vector  $\mathbf{f}(t)$  are continuous over  $(t_L, t_R)$ ,
- the initial time  $t_I$  is in  $(t_L, t_R)$ ,
- an entry of either the coefficient matrix or the forcing vector is undefined at each of  $t = t_L$  and  $t = t_R$ .

You can do this because the first two bullets along with the Basic Existence and Uniqueness Theorem imply that the interval of definition will be at least  $(t_L, t_R)$ , while the last two bullets along with our definition of solution imply that the interval of definition can be no bigger than  $(t_L, t_R)$  because the equation breaks down at  $t = t_L$  and  $t = t_R$ . This argument works when  $t_L = -\infty$  or  $t_R = \infty$ .

**2.2. Homogeneous Systems.** Just as with higher-order linear equations, the key to solving a first-order linear system (2.2) is understanding how to solve its associated homogeneous system

$$(2.4) \quad \frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}.$$

We will assume throughout this section that the coefficient matrix  $\mathbf{A}(t)$  is continuous over an interval  $(t_L, t_R)$ , so that Theorem 1.1 can be applied. We will exploit the following property of homogeneous systems.

**Theorem 2.2. (Linear Superposition).** If  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  are solutions of system (2.4) then so is  $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$  for any values of the constants  $c_1$  and  $c_2$ . More generally, if  $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_m(t)$  are  $m$  solutions of system (2.4) then so is the linear combination

$$(2.5) \quad \mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_m\mathbf{x}_m(t),$$

for any values of the constants  $c_1, c_2, \dots, c_m$ .

**Remark.** Here  $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_m(t)$  denote  $m$  different vector-valued solutions of the system (2.4), and should not be confused with  $x_1(t), x_2(t), \dots, x_m(t)$ , which denote the first  $m$  entries of the vector-valued function  $\mathbf{x}(t)$ .

**Reason.** Because  $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_m(t)$  solve (2.4), a direct calculation starting from the linear combination (2.5) shows that

$$\begin{aligned} \frac{d\mathbf{x}}{dt}(t) &= \frac{d}{dt}(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_m\mathbf{x}_m(t)) \\ &= c_1\frac{d\mathbf{x}_1}{dt}(t) + c_2\frac{d\mathbf{x}_2}{dt}(t) + \dots + c_m\frac{d\mathbf{x}_m}{dt}(t) \\ &= c_1\mathbf{A}(t)\mathbf{x}_1(t) + c_2\mathbf{A}(t)\mathbf{x}_2(t) + \dots + c_m\mathbf{A}(t)\mathbf{x}_m(t) \\ &= \mathbf{A}(t)(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_m\mathbf{x}_m(t)) \\ &= \mathbf{A}(t)\mathbf{x}(t). \end{aligned}$$

Therefore  $\mathbf{x}(t)$  given by the linear combination (2.5) solves system (2.4).  $\square$

**Remark.** This theorem states that any linear combination of solutions of (2.4) is also a solution of (2.4). It thereby provides a way to construct a whole family of solutions from a finite number of them.

Now consider the initial-value problem

$$(2.6) \quad \frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{x}(t_I) = \mathbf{x}^I.$$

Suppose we know  $n$  “different” solutions of (2.4),  $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ . It is natural to ask if we can construct the solution of the initial-value problem (2.6) as a linear combination of  $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ . Set

$$(2.7) \quad \mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_n\mathbf{x}_n(t).$$

By the superposition theorem this is a solution of (2.4). We only have to check that values of  $c_1, c_2, \dots, c_n$  can be found so that  $\mathbf{x}(t)$  will also satisfy the initial conditions in (2.6) — namely, so that

$$\mathbf{x}^I = \mathbf{x}(t_I) = c_1\mathbf{x}_1(t_I) + c_2\mathbf{x}_2(t_I) + \dots + c_n\mathbf{x}_n(t_I) = \mathbf{\Psi}(t_I)\mathbf{c},$$

where  $\mathbf{c}$  is the  $n \times 1$  column vector given by

$$\mathbf{c} = (c_1 \ c_2 \ \dots \ c_n)^T,$$

while  $\mathbf{\Psi}(t_I)$  is the  $n \times n$  matrix given by

$$\mathbf{\Psi}(t_I) = (\mathbf{x}_1(t_I) \ \mathbf{x}_2(t_I) \ \dots \ \mathbf{x}_n(t_I)).$$

This notation indicates that the  $k^{\text{th}}$  column of  $\mathbf{\Psi}(t_I)$  is the column vector  $\mathbf{x}_k(t_I)$ . In this notation the question becomes whether there is a vector  $\mathbf{c}$  such that

$$\mathbf{\Psi}(t_I)\mathbf{c} = \mathbf{x}^I, \quad \text{for every } \mathbf{x}^I.$$

This linear algebraic system will have a solution for every  $\mathbf{x}^I$  if and only if the matrix  $\mathbf{\Psi}(t_I)$  is invertible, in which case the solution is unique and is given by

$$\mathbf{c} = \mathbf{\Psi}(t_I)^{-1}\mathbf{x}^I.$$

Of course, the matrix  $\mathbf{\Psi}(t_I)$  is invertible if and only if  $\det(\mathbf{\Psi}(t_I)) \neq 0$ .

**2.3. Wronskians and Fundamental Matrices.** Given any set of  $n$  solutions  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$ ,  $\dots$ ,  $\mathbf{x}_n(t)$  to the homogeneous equation (2.4), we define its *Wronskian* by

$$(2.8) \quad W[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n](t) = \det \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \dots & \mathbf{x}_n(t) \end{pmatrix} .$$

The *Abel Theorem* for first-order systems is

$$(2.9) \quad \frac{d}{dt} W[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n](t) = \operatorname{tr}(\mathbf{A}(t)) W[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n](t) ,$$

where  $\operatorname{tr}(\mathbf{A}(t))$  denotes the *trace* of  $\mathbf{A}(t)$ , which is given by

$$(2.10) \quad \operatorname{tr}(\mathbf{A}(t)) = a_{11}(t) + a_{22}(t) + \dots + a_{nn}(t) .$$

Upon integrating the first-order linear equation (2.9) we see that

$$(2.11) \quad W[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n](t) = W[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n](t_I) \exp \left( \int_{t_I}^t \operatorname{tr}(\mathbf{A}(s)) ds \right) .$$

As was the case for higher-order equations, this shows that if the Wronkian is nonzero somewhere then it is nonzero everywhere, and that if it is zero somewhere, it is zero everywhere!

Again analogous to the case for higher-order equations, we have the following definitions.

**Definition.** A set of  $n$  solutions  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$ ,  $\dots$ ,  $\mathbf{x}_n(t)$  to the  $n$ -dimensional homogeneous linear system (2.4) called *fundamental* if its Wronskian is nonzero. Then the family

$$(2.12) \quad \mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \dots + c_n \mathbf{x}_n(t)$$

is called a *general solution* of system (2.4).

However, now we introduce a new concept for first-order systems.

**Definition.** If  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$ ,  $\dots$ ,  $\mathbf{x}_n(t)$  is a fundamental set of solutions to system (2.4) then the  $n \times n$  matrix-valued function

$$(2.13) \quad \mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \dots & \mathbf{x}_n(t) \end{pmatrix}$$

is called a *fundamental matrix* for system (2.4).

Some basic facts about fundamental matrices are as follows.

**Fact.** Let  $\mathbf{\Psi}(t)$  be a fundamental matrix for system (2.4). Then

- $\mathbf{\Psi}(t)$  satisfies

$$(2.14) \quad \mathbf{\Psi}' = \mathbf{A}(t)\mathbf{\Psi} , \quad \det(\mathbf{\Psi}(t)) \neq 0 ;$$

- A general solution of system (2.4) is

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c} ;$$

**Reason.** By (2.13) we see that

$$\begin{aligned}\Psi'(t) &= (\mathbf{x}_1(t) \ \mathbf{x}_2(t) \ \cdots \ \mathbf{x}_n(t))' = (\mathbf{x}'_1(t) \ \mathbf{x}'_2(t) \ \cdots \ \mathbf{x}'_n(t)) \\ &= (\mathbf{A}(t)\mathbf{x}_1(t) \ \mathbf{A}(t)\mathbf{x}_2(t) \ \cdots \ \mathbf{A}(t)\mathbf{x}_n(t)) \\ &= \mathbf{A}(t) (\mathbf{x}_1(t) \ \mathbf{x}_2(t) \ \cdots \ \mathbf{x}_n(t)) \\ &= \mathbf{A}(t)\Psi(t).\end{aligned}$$

Also by (2.13) we see that

$$\begin{aligned}\det(\Psi(t)) &= \det(\mathbf{x}_1(t) \ \mathbf{x}_2(t) \ \cdots \ \mathbf{x}_n(t)) \\ &= W[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n](t) \neq 0.\end{aligned}$$

It should be clear from (2.13) that the general solution given by (2.12) can be expressed as  $\mathbf{x}(t) = \Psi(t)\mathbf{c}$ .

**Example.** The vector-valued functions

$$\mathbf{x}_1(t) = e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

are solutions of the differential system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Construct a general solution and a fundamental matrix for this system.

**Solution.** It is easy to check that  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  are each solutions to the differential system. Because

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix} = -2e^{6t},$$

we see that  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  comprise a fundamental set of solutions to the system. Therefore a fundamental matrix is given by

$$\Psi(t) = (\mathbf{x}_1(t) \ \mathbf{x}_2(t)) = \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix},$$

while a general solution is given by

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} = \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{5t} + c_2 e^t \\ c_1 e^{5t} - c_2 e^t \end{pmatrix}.$$

Alternatively, we can construct a general solution as

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) = c_1 \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix} + c_2 \begin{pmatrix} e^t \\ -e^t \end{pmatrix} = \begin{pmatrix} c_1 e^{5t} + c_2 e^t \\ c_1 e^{5t} - c_2 e^t \end{pmatrix}.$$

**Example.** The vector-valued functions

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 + t^3 \\ t \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix},$$

are solutions of the differential system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Construct a general solution and a fundamental matrix for this system.

**Solution.** It is easy to check that  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  are each solutions to the differential system. Because

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} = 1,$$

we see that  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  comprise a fundamental set of solutions to the system. Therefore a fundamental matrix is given by

$$\mathbf{\Psi}(t) = (\mathbf{x}_1(t) \quad \mathbf{x}_2(t)) = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix},$$

while a general solution is given by

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c} = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1(1 + t^3) + c_2t^2 \\ c_1t + c_2 \end{pmatrix}.$$

Alternatively, we can construct a general solution as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} 1 + t^3 \\ t \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1(1 + t^3) + c_2t^2 \\ c_1t + c_2 \end{pmatrix}.$$

**Remark.** The solutions  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  were given to you in the problems above. Sections 3 and 4 will present methods by which we can construct a fundamental set of solutions (and therefore a fundamental matrix) for any homogeneous system with a constant coefficient matrix. For systems with a variable coefficient matrix you will always be given solutions.

**Remark.** Any matrix-valued function  $\mathbf{\Psi}(t)$  such that  $\det(\mathbf{\Psi}(t)) \neq 0$  over some time interval  $(t_L, t_R)$  is a fundamental matrix for the first-order differential system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}, \quad \text{where } \mathbf{A}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1}.$$

This can be seen by multiplying (2.14) on the left by  $\mathbf{\Psi}(t)^{-1}$ .

**Example.** Find a first-order differential system such that the vector-valued functions

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 + t^3 \\ t \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix},$$

comprise a fundamental set of solutions.

**Solution.** Set

$$\mathbf{\Psi}(t) = (\mathbf{x}_1(t) \quad \mathbf{x}_2(t)) = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix}.$$

Because  $\det(\Psi(t)) = 1$ , we see that  $\Psi(t)$  is invertible. Set

$$\begin{aligned} \mathbf{A}(t) &= \Psi'(t)\Psi(t)^{-1} = \begin{pmatrix} 3t^2 & 2t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+t^3 & t^2 \\ t & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3t^2 & 2t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t^2 \\ -t & 1+t^3 \end{pmatrix} = \begin{pmatrix} t^2 & 2t-t^4 \\ 1 & -t^2 \end{pmatrix}. \end{aligned}$$

Therefore  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  are a fundamental set of solutions for the differential system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t-t^4 \\ 1 & -t^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

**2.4. Natural Fundamental Matrices.** Consider the initial-value problem

$$(2.15) \quad \mathbf{x}' = \mathbf{A}(t)\mathbf{x}(t), \quad \mathbf{x}(t_I) = \mathbf{x}^I.$$

Let  $\Psi(t)$  be any fundamental matrix for this system. Then a general solution of the system is given by

$$\mathbf{x}(t) = \Psi(t)\mathbf{c}.$$

By imposing the initial condition from (2.15) we see that

$$\mathbf{x}^I = \mathbf{x}(t_I) = \Psi(t_I)\mathbf{c}.$$

Because  $\det(\Psi(t_I)) \neq 0$ , the matrix  $\Psi(t_I)$  is invertible and we can solve for  $\mathbf{c}$  as

$$\mathbf{c} = \Psi(t_I)^{-1}\mathbf{x}^I.$$

Hence, the solution of the initial-value problem is

$$(2.16) \quad \mathbf{x}(t) = \Psi(t)\Psi(t_I)^{-1}\mathbf{x}^I.$$

Now let  $\Phi(t)$  be the matrix-valued function defined by

$$(2.17) \quad \Phi(t) = \Psi(t)\Psi(t_I)^{-1}.$$

If we differentiate  $\Phi(t)$  and use the fact that  $\Psi(t)$  is a fundamental matrix for system (2.15) we see that

$$\Phi'(t) = (\Psi(t)\Psi(t_I)^{-1})' = \Psi'(t)\Psi(t_I)^{-1} = \mathbf{A}(t)\Psi(t)\Psi(t_I)^{-1} = \mathbf{A}(t)\Phi(t).$$

Moreover, from (2.17) we see that

$$\Phi(t_I) = \Psi(t_I)\Psi(t_I)^{-1} = \mathbf{I}.$$

Therefore  $\Phi(t)$  is the solution of the matrix-valued initial-value problem

$$(2.18) \quad \Phi' = \mathbf{A}(t)\Phi, \quad \Phi(t_I) = \mathbf{I}.$$

This shows three things.

1.  $\Phi(t)$  as a function of  $t$  is a fundamental matrix for system (2.15);
2.  $\Phi(t)$  is uniquely determined by the matrix-valued initial-value problem (2.18);
3.  $\Phi(t)$  is independent of our original choice of fundamental matrix  $\Psi(t)$  that was used to construct it in (2.17).

We call  $\Phi(t)$  the *natural fundamental matrix* associated with the initial time  $t_I$ .

Just like it was easy to express the solution of an initial-value problem for a higher-order equation in terms of its associated natural fundamental sets of solutions, we express the solution of the initial-value problem (2.15) in terms of its associated natural fundamental matrix as simply

$$(2.19) \quad \mathbf{x}(t) = \Phi(t)\mathbf{x}^I.$$

Given any fundamental matrix  $\Psi(t)$ , we construct the natural fundamental matrix associated with the initial time  $t_I$  by formula (2.17).

**Example.** Construct the natural fundamental matrix associated with the initial time 0 for the system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Use it to solve the initial-value problem with the initial conditions  $x_1(0) = 4$  and  $x_2(0) = -2$ .

**Solution.** We have already seen that a fundamental matrix for this system is

$$\Psi(t) = \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix}.$$

By formula (2.17) the natural fundamental matrix associated with the initial time 0 is

$$\begin{aligned} \Phi(t) &= \Psi(t)\Psi(0)^{-1} = \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix} \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{5t} + e^t & e^{5t} - e^t \\ e^{5t} - e^t & e^{5t} + e^t \end{pmatrix}. \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}^I = \frac{1}{2} \begin{pmatrix} e^{5t} + e^t & e^{5t} - e^t \\ e^{5t} - e^t & e^{5t} + e^t \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{5t} + 3e^t \\ e^{5t} - 3e^t \end{pmatrix}.$$

**Example.** Construct the natural fundamental matrix associated with the initial time 1 for the system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Use it to solve the initial-value problem with the initial conditions  $x_1(1) = 3$  and  $x_2(1) = 0$ .

**Solution.** We have already seen that a fundamental matrix for this system is

$$\Psi(t) = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix}.$$

By formula (2.17) the natural fundamental matrix associated with the initial time 1 is

$$\begin{aligned} \Phi(t) &= \Psi(t)\Psi(1)^{-1} = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 + t^3 - t^2 & -1 - t^3 + 2t^2 \\ t - 1 & 2 - t \end{pmatrix}. \end{aligned}$$



Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}^I = \begin{pmatrix} 1 + t^3 - t^2 & -1 - t^3 + 2t^2 \\ t - 1 & 2 - t \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 + t^3 - t^2 \\ t - 1 \end{pmatrix}.$$

**2.5. Nonhomogeneous Systems and Green Matrices.** We now consider the non-homogenous first-order linear system

$$(2.20) \quad \mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t).$$

If  $\mathbf{x}_P(t)$  is a particular solution of this system and  $\Psi(t)$  is a fundamental matrix of the associated homogeneous system then a general solution of system (2.20) is

$$\mathbf{x}(t) = \mathbf{x}_H(t) + \mathbf{x}_P(t),$$

where  $\mathbf{x}_H(t)$  is the general solution of the associated homogeneous problem given by

$$(2.21) \quad \mathbf{x}_H(t) = \Psi(t)\mathbf{c}.$$

Recall that if we know a fundamental set of solutions to the associated homogeneous differential equation then we can use either Variations of Parameters or general Green Functions to construct a particular solution to a nonhomogeneous  $n^{\text{th}}$ -order linear equation in terms of  $n$  integrals. Here we show that a similar thing is true for the non-homogenous first-order linear system (2.20). Specifically, if we know a fundamental matrix  $\Psi(t)$  for the associated homogeneous system then we can construct a particular solution to the  $n$  dimensional nonhomogeneous linear equation in terms of  $n$  integrals.

We begin with the analog of the method of Variation of Parameters for the non-homogeneous first-order linear system (2.20). We will assume that  $\mathbf{A}(t)$  and  $\mathbf{f}(t)$  are continuous over an interval  $(t_L, t_R)$ . We also will assume that we know a fundamental matrix  $\Psi(t)$  of the associated homogeneous system. This matrix will be continuously differentiable over  $(t_L, t_R)$  and satisfy

$$\Psi'(t) = \mathbf{A}(t)\Psi(t), \quad \det(\Psi(t)) \neq 0.$$

Because  $\mathbf{x}_H(t)$  has the form (2.21), we *seek* a particular solution in the form

$$(2.22) \quad \mathbf{x}_P(t) = \Psi(t)\mathbf{u}(t),$$

where  $\mathbf{u}(t)$  is a vector-valued function. By differentiation we see that

$$\begin{aligned} \mathbf{x}'_P(t) &= (\Psi(t)\mathbf{u}(t))' = \Psi'(t)\mathbf{u}(t) + \Psi(t)\mathbf{u}'(t) \\ &= \mathbf{A}(t)\Psi(t)\mathbf{u}(t) + \Psi(t)\mathbf{u}'(t) \\ &= \mathbf{A}(t)\mathbf{x}_P(t) + \Psi(t)\mathbf{u}'(t). \end{aligned}$$

By comparing the right-hand side of this equation with the right-hand side of equation (2.20), we see that  $\mathbf{x}_P(t)$  will solve (2.20) if  $\mathbf{u}(t)$  satisfies

$$\Psi(t)\mathbf{u}'(t) = \mathbf{f}(t).$$

Because  $\Psi(t)$  is invertible, we find that

$$(2.23) \quad \mathbf{u}'(t) = \Psi(t)^{-1}\mathbf{f}(t).$$

If  $\mathbf{u}_P(t)$  is a primitive of the right-hand side above then a general solution of this system has the form

$$\mathbf{u}(t) = \mathbf{c} + \mathbf{u}_P(t).$$

When this solution is placed into the form (2.22), we find that a particular solution is given by

$$(2.24) \quad \mathbf{x}_P(t) = \mathbf{\Psi}(t)\mathbf{u}_P(t).$$

Now let  $t_I$  be any initial time in  $(t_L, t_R)$  and consider the initial-value problem

$$(2.25) \quad \mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(t_I) = \mathbf{x}^I.$$

If we take the particular solution of (2.23) given by

$$\mathbf{u}_P(t) = \int_{t_I}^t \mathbf{\Psi}(s)^{-1}\mathbf{f}(s) \, ds,$$

then (2.24) becomes

$$(2.26) \quad \mathbf{x}_P(t) = \mathbf{\Psi}(t) \int_{t_I}^t \mathbf{\Psi}(s)^{-1}\mathbf{f}(s) \, ds.$$

The solution of the initial-value problem is then

$$(2.27) \quad \mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_I)^{-1}\mathbf{x}^I + \mathbf{\Psi}(t) \int_{t_I}^t \mathbf{\Psi}(s)^{-1}\mathbf{f}(s) \, ds.$$

We define the *Green matrix*  $\mathbf{G}(t, s)$  by

$$(2.28) \quad \mathbf{G}(t, s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1}.$$

Then we can recast (2.27) as

$$(2.29) \quad \mathbf{x}(t) = \mathbf{G}(t, t_I)\mathbf{x}^I + \int_{t_I}^t \mathbf{G}(t, s) \mathbf{f}(s) \, ds.$$

In particular, the particular solution of (2.27) that satisfies  $\mathbf{x}(t_I) = \mathbf{0}$  is given by

$$(2.30) \quad \mathbf{x}(t) = \int_{t_I}^t \mathbf{G}(t, s) \mathbf{f}(s) \, ds.$$

The Green matrix has the property that for every  $t_I$  in  $(t_L, t_R)$  the natural fundamental matrix associated with  $t_I$  is given by  $\mathbf{\Phi}(t) = \mathbf{G}(t, t_I)$ .

**Example.** Construct the Green matrix for the system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{1 + e^{-2t}} \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

**Solution.** We have seen that a fundamental matrix associated with this system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix}.$$

Then by formula (2.28) the Green matrix is given by

$$\begin{aligned} \mathbf{G}(t, s) &= \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} = \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix} \begin{pmatrix} e^{5s} & e^s \\ e^{5s} & -e^s \end{pmatrix}^{-1} \\ &= \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix} \frac{1}{-2e^{6s}} \begin{pmatrix} -e^s & -e^s \\ -e^{5s} & e^{5s} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix} \begin{pmatrix} e^{-5s} & e^{-5s} \\ e^{-s} & -e^{-s} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{5(t-s)} + e^{t-s} & e^{5(t-s)} - e^{t-s} \\ e^{5(t-s)} - e^{t-s} & e^{5(t-s)} + e^{t-s} \end{pmatrix}. \end{aligned}$$

**Example.** Construct the Green matrix for the system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e^t \\ t \end{pmatrix}.$$

**Solution.** We have seen that a fundamental matrix associated with this system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix}.$$

Then by formula (2.28) the Green matrix is given by

$$\begin{aligned} \mathbf{G}(t, s) &= \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 + s^3 & s^2 \\ s & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & -s^2 \\ -s & 1 + s^3 \end{pmatrix} = \begin{pmatrix} 1 + t^3 - t^2s & t^2 + t^2s^3 - s^2 - t^3s^2 \\ t - s & 1 + s^3 - ts^2 \end{pmatrix}. \end{aligned}$$

## B. APPENDIX: VECTORS AND MATRICES

An  $m \times n$  matrix  $\mathbf{A}$  consists of a rectangular array of entries arranged in  $m$  rows and  $n$  columns

$$(2.31) \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{jk}).$$

We call  $a_{jk}$  the  $jk$ -entry of  $\mathbf{A}$ ,  $m$  the row-dimension of  $\mathbf{A}$ ,  $n$  the column-dimension of  $\mathbf{A}$ , and  $m \times n$  the dimensions of  $\mathbf{A}$ . The entries of a matrix can be drawn from any set, but in this course they will be numbers. Special kinds of matrices include:

- $1 \times m$  matrices are called *row vectors*;
- $n \times 1$  matrices are called *column vectors*;
- $n \times n$  matrices are called *square matrices*.

Two  $m \times n$  matrices  $\mathbf{A} = (a_{jk})$  and  $\mathbf{B} = (b_{jk})$  are said to be *equal* if  $a_{jk} = b_{jk}$  for every  $j = 1, 2, \dots, m$  and  $k = 1, 2, \dots, n$ , in which case we write  $\mathbf{A} = \mathbf{B}$ . We will use  $\mathbf{0}$  to denote any matrix or vector that has every entry equal to zero. A matrix or vector is said to be *nonzero* if at least one of its entries is not equal to zero.

**B.1. Vector and Matrix Operations.**

*Matrix Addition.* Two  $m \times n$  matrices  $\mathbf{A} = (a_{jk})$  and  $\mathbf{B} = (b_{jk})$  can be added to create a new  $m \times n$  matrix sum  $\mathbf{A} + \mathbf{B}$ , called the *sum* of  $\mathbf{A}$  and  $\mathbf{B}$ , defined by

$$\mathbf{A} + \mathbf{B} = (a_{jk} + b_{jk}).$$

If matrices  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  have the same dimensions then matrix addition satisfies

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \mathbf{B} + \mathbf{A} && \text{—commutativity,} \\ (\mathbf{A} + \mathbf{B}) + \mathbf{C} &= \mathbf{A} + (\mathbf{B} + \mathbf{C}) && \text{—associativity,} \\ \mathbf{A} + \mathbf{0} &= \mathbf{0} + \mathbf{A} = \mathbf{A} && \text{—additive identity,} \\ \mathbf{A} + (-\mathbf{A}) &= (-\mathbf{A}) + \mathbf{A} = \mathbf{0} && \text{—additive inverse.} \end{aligned}$$

Here the matrix  $-\mathbf{A}$  is defined by  $-\mathbf{A} = (-a_{jk})$  when  $\mathbf{A} = (a_{jk})$ . We define *matrix subtraction* by  $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$ .

*Scalar Multiplication.* A number  $\alpha$  and an  $m \times n$  matrix  $\mathbf{A} = (a_{jk})$  can be multiplied to create a new  $m \times n$  matrix  $\alpha\mathbf{A}$ , called the *multiple* of  $\mathbf{A}$  by  $\alpha$ , defined by

$$\alpha\mathbf{A} = (\alpha a_{jk}).$$

If matrices  $\mathbf{A}$  and  $\mathbf{B}$  have the same dimensions then scalar multiplication satisfies

$$\begin{aligned} \alpha(\beta\mathbf{A}) &= (\alpha\beta)\mathbf{A} && \text{—associativity,} \\ \alpha(\mathbf{A} + \mathbf{B}) &= \alpha\mathbf{A} + \alpha\mathbf{B} && \text{—distributivity over matrix addition,} \\ (\alpha + \beta)\mathbf{A} &= \alpha\mathbf{A} + \beta\mathbf{A} && \text{—distributivity over scalar addition,} \\ 1\mathbf{A} &= \mathbf{A}, \quad -1\mathbf{A} = -\mathbf{A} && \text{—scalar identity,} \\ 0\mathbf{A} &= \mathbf{0}, \quad \alpha\mathbf{0} = \mathbf{0} && \text{—scalar multiplicative nullity.} \end{aligned}$$

*Matrix Multiplication.* An  $l \times m$  matrix  $\mathbf{A}$  and an  $m \times n$  matrix  $\mathbf{B}$  can be multiplied to create a new  $l \times n$  matrix  $\mathbf{AB}$ , called the *product* of  $\mathbf{A}$  and  $\mathbf{B}$ , defined by

$$\mathbf{AB} = (c_{ik}), \quad \text{where} \quad c_{ik} = \sum_{j=1}^m a_{ij}b_{jk}.$$

**Remark.** Notice that for some matrices  $\mathbf{A}$  and  $\mathbf{B}$ , depending only on their dimensions, neither  $\mathbf{AB}$  nor  $\mathbf{BA}$  exist; for others exactly one of  $\mathbf{AB}$  or  $\mathbf{BA}$  exists; while for others both  $\mathbf{AB}$  and  $\mathbf{BA}$  exist.

**Example.** For the matrices

$$\mathbf{A} = \begin{pmatrix} 3 & 5 + i6 \\ 2 & i3 \\ 2 + i3 & 2 - i \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 - i5 & -4 \\ -1 & 6 + i \end{pmatrix},$$

the product  $\mathbf{AB}$  exists with

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 3 \cdot (2 - i5) + (5 + i6) \cdot (-1) & 3 \cdot (-4) + (5 + i6) \cdot (6 + i) \\ 2 \cdot (2 - i5) + (i3) \cdot (-1) & 2 \cdot (-4) + (i3) \cdot (6 + i) \\ (2 + i3) \cdot (2 - i5) + (2 - i) \cdot (-1) & (2 + i3) \cdot (-4) + (2 - i) \cdot (6 + i) \end{pmatrix} \\ &= \begin{pmatrix} 12 - i20 & 12 + i41 \\ 7 - i10 & -11 + i18 \\ 18 - i6 & 5 - i16 \end{pmatrix}, \end{aligned}$$

while the product  $\mathbf{BA}$  does not exist.

**Remark.** Notice that if  $\mathbf{A}$  and  $\mathbf{B}$  are  $n \times n$  then  $\mathbf{AB}$  and  $\mathbf{BA}$  both exist and are  $n \times n$ , but in general

$$\mathbf{AB} \neq \mathbf{BA}!$$

**Example.** For the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -3 & 9 \\ 1 & -3 \end{pmatrix},$$

we see that

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} -3 & 9 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathbf{BA} &= \begin{pmatrix} -3 & 9 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 15 & 45 \\ -5 & -15 \end{pmatrix}, \end{aligned}$$

whereby  $\mathbf{AB} \neq \mathbf{BA}$ .

**Remark.** The above example also shows that

$$\mathbf{AB} = \mathbf{0} \quad \text{does not imply that either } \mathbf{A} = \mathbf{0} \text{ or } \mathbf{B} = \mathbf{0}!$$

If  $\alpha$  is a number and  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are matrices that have the dimensions indicated on the left then matrix multiplication satisfies

$$\begin{array}{llll}
 \mathbf{A} & \mathbf{B} & \mathbf{C} & \\
 l \times m & m \times n & & (\alpha\mathbf{A})\mathbf{B} = \alpha(\mathbf{AB}) \quad \text{---associativity,} \\
 l \times m & m \times n & n \times k & (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \quad \text{---associativity,} \\
 l \times m & m \times n & m \times n & \mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \quad \text{---left distributivity,} \\
 l \times m & l \times m & m \times n & (\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC} \quad \text{---right distributivity.}
 \end{array}$$

An *identity matrix* is a square matrix  $\mathbf{I}$  in the form

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We will use  $\mathbf{I}$  to denote any identity matrix; the dimensions of  $\mathbf{I}$  will always be clear from the context. Identity matrices have the property for any  $m \times n$  matrix  $\mathbf{A}$

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A} \quad \text{---multiplicative identity.}$$

Notice that the first  $\mathbf{I}$  above is  $m \times m$  while the second is  $n \times n$ .

Similarly, zero matrices have the property for any  $m \times n$  matrix  $\mathbf{A}$

$$\mathbf{0A} = \mathbf{0}, \quad \mathbf{A0} = \mathbf{0} \quad \text{---multiplicative nullity.}$$

If the first  $\mathbf{0}$  is  $l \times m$  then the second is  $l \times n$ . If the third  $\mathbf{0}$  is  $n \times k$  then the fourth is  $m \times k$ .

*Matrix Conjugate.* The conjugate of the  $m \times n$  matrix  $\mathbf{A}$  given by (2.31) is the  $m \times n$  matrix  $\overline{\mathbf{A}}$  given by

$$\overline{\mathbf{A}} = (\overline{a_{jk}}).$$

If  $\overline{\mathbf{A}} = \mathbf{A}$  then each entry of  $\mathbf{A}$  is real and  $\mathbf{A}$  is called a *real matrix*. If  $\overline{\mathbf{A}} = -\mathbf{A}$  then each entry of  $\mathbf{A}$  is imaginary and  $\mathbf{A}$  is called a *imaginary matrix*.

If  $\alpha$  is a number and  $\mathbf{A}$  and  $\mathbf{B}$  are matrices that have the dimensions indicated on the left then matrix conjugate satisfies

$$\begin{array}{llll}
 \mathbf{A} & \mathbf{B} & & \\
 m \times n & m \times n & \overline{(\mathbf{A} + \mathbf{B})} = \overline{\mathbf{A}} + \overline{\mathbf{B}}, \\
 m \times n & & \overline{(\alpha\mathbf{A})} = \overline{\alpha}\overline{\mathbf{A}}, \\
 l \times m & m \times n & \overline{\mathbf{AB}} = \overline{\mathbf{A}}\overline{\mathbf{B}} \quad \text{(note no flip),} \\
 m \times n & & \overline{(\overline{\mathbf{A}})} = \mathbf{A}.
 \end{array}$$

*Matrix Transpose.* The transpose of the  $m \times n$  matrix  $\mathbf{A}$  given by (2.31) is the  $n \times m$  matrix  $\mathbf{A}^T$  given by

$$\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

If  $\alpha$  is a number and  $\mathbf{A}$  and  $\mathbf{B}$  are matrices that have the dimensions indicated on the left then matrix transpose satisfies

$$\begin{array}{ll} \mathbf{A} & \mathbf{B} \\ m \times n & m \times n & (\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T, \\ m \times n & & (\alpha\mathbf{A})^T = \alpha\mathbf{A}^T, \\ l \times m & m \times n & (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \quad (\text{note flip}), \\ m \times n & & (\mathbf{A}^T)^T = \mathbf{A}. \end{array}$$

*Hermitian Transpose.* The Hermitian transpose of the  $m \times n$  matrix  $\mathbf{A}$  given by (2.31) is the  $n \times m$  matrix  $\mathbf{A}^* = \overline{\mathbf{A}^T} = \overline{\mathbf{A}}^T$ . If  $\alpha$  is a number and  $\mathbf{A}$  and  $\mathbf{B}$  are matrices that have the dimensions indicated on the left then Hermitian transpose satisfies

$$\begin{array}{ll} \mathbf{A} & \mathbf{B} \\ m \times n & m \times n & (\mathbf{A} + \mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*, \\ m \times n & & (\alpha\mathbf{A})^* = \overline{\alpha}\mathbf{A}^*, \\ l \times m & m \times n & (\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^* \quad (\text{note flip}), \\ m \times n & & (\mathbf{A}^*)^* = \mathbf{A}. \end{array}$$

**Examples.** For the matrices

$$\mathbf{A} = \begin{pmatrix} 3 & 5 + i6 \\ 2 & i3 \\ 2 + i3 & 2 - i \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 - i5 & -4 \\ -1 & 6 + i \end{pmatrix},$$

we have

$$\begin{array}{ll} \overline{\mathbf{A}} = \begin{pmatrix} 3 & 5 - i6 \\ 2 & -i3 \\ 2 - i3 & 2 + i \end{pmatrix}, & \overline{\mathbf{B}} = \begin{pmatrix} 2 + i5 & -4 \\ -1 & 6 - i \end{pmatrix}, \\ \mathbf{A}^T = \begin{pmatrix} 3 & 2 & 2 + i3 \\ 5 + i6 & i3 & 2 - i \end{pmatrix}, & \mathbf{B}^T = \begin{pmatrix} 2 + i5 & -1 \\ -4 & 6 - i \end{pmatrix}, \\ \mathbf{A}^* = \begin{pmatrix} 3 & 2 & 2 - i3 \\ 5 - i6 & -i3 & 2 + i \end{pmatrix}, & \mathbf{B}^* = \begin{pmatrix} 2 - i5 & -1 \\ -4 & 6 + i \end{pmatrix}, \end{array}$$

**B.2. Invertibility and Inverses.** An  $n \times n$  matrix  $\mathbf{A}$  is said to be *invertible* if there exists another  $n \times n$  matrix  $\mathbf{B}$  such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I},$$

in which case  $\mathbf{B}$  is said to be an *inverse* of  $\mathbf{A}$ .

**Fact.** A matrix can have at most one inverse.

**Reason.** Suppose that  $\mathbf{B}$  and  $\mathbf{C}$  are inverses of  $\mathbf{A}$ . Then

$$\mathbf{B} = \mathbf{BI} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}.$$

If  $\mathbf{A}$  is invertible then its unique inverse is denoted  $\mathbf{A}^{-1}$ .

**Fact.** If  $\mathbf{A}$  and  $\mathbf{B}$  are invertible  $n \times n$  matrices and  $\alpha \neq 0$  then

- $\alpha\mathbf{A}$  is invertible with  $(\alpha\mathbf{A})^{-1} = \frac{1}{\alpha}\mathbf{A}^{-1}$ ;
- $\mathbf{AB}$  is invertible with  $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$  (notice the flip);
- $\mathbf{A}^{-1}$  is invertible with  $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ ;
- $\overline{\mathbf{A}}$  is invertible with  $(\overline{\mathbf{A}})^{-1} = \overline{\mathbf{A}^{-1}}$ ;
- $\mathbf{A}^T$  is invertible with  $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ ;
- $\mathbf{A}^*$  is invertible with  $(\mathbf{A}^*)^{-1} = (\mathbf{A}^{-1})^*$ .

**Reason.** Each of these facts can be checked by direct calculation. For example, the second fact is checked by

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = ((\mathbf{AB})\mathbf{B}^{-1})\mathbf{A}^{-1} = (\mathbf{A}(\mathbf{BB}^{-1}))\mathbf{A}^{-1} = (\mathbf{AI})\mathbf{A}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}.$$

**Fact.** If  $\mathbf{A}$  and  $\mathbf{B}$  are  $n \times n$  matrices and  $\mathbf{AB}$  is invertible then both  $\mathbf{A}$  and  $\mathbf{B}$  are invertible with

$$\mathbf{A}^{-1} = \mathbf{B}(\mathbf{AB})^{-1}, \quad \mathbf{B}^{-1} = (\mathbf{AB})^{-1}\mathbf{A}.$$

**Reason.** Each of these facts can be checked by direct calculation.

**Fact.** If  $\mathbf{A}$  is invertible and  $\mathbf{AB} = \mathbf{0}$  then  $\mathbf{B} = \mathbf{0}$ .

**Reason.**

$$\mathbf{B} = \mathbf{IB} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}.$$

**Fact.** Not all nonzero square matrices are invertible.

**Reason.** Earlier we gave an example of two nonzero matrices  $\mathbf{A}$  and  $\mathbf{B}$  such that  $\mathbf{AB} = \mathbf{0}$ . The previous fact then implies that  $\mathbf{A}$  is not invertible.



*Determinants and Invertibility.* The invertability of a square matrix is characterized by its determinant.

**Fact.** A matrix  $\mathbf{A}$  is invertible if and only if  $\det(\mathbf{A}) \neq 0$ .

**Reason.** This fact is proved in Linear Algebra courses. We will prove it for the special case of  $2 \times 2$  matrices in the following example.

**Example.** For  $2 \times 2$  matrices the inverse is easy to compute when it exists. If

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then  $\det(\mathbf{A}) = ad - bc$ . If  $\det(\mathbf{A}) = ad - bc \neq 0$  then  $\mathbf{A}$  is invertible with

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This result follows from the calculation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & -ab + ba \\ -cd + dc & -cb + da \end{pmatrix} = (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det(\mathbf{A})\mathbf{I}.$$

This calculation also shows that if  $\det(\mathbf{A}) = ad - bc = 0$  then  $\mathbf{A}$  is not invertible.

The following is a very important fact about determinants.

**Fact.** If  $\mathbf{A}$  and  $\mathbf{B}$  are  $n \times n$  matrices then  $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$ .

**Reason.** We can prove this fact for  $2 \times 2$  matrices by direct calculation. If

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

then

$$\begin{aligned} \det(\mathbf{AB}) &= \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) = \det\left(\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}\right) \\ &= (ae + bg)(cf + dh) - (ce + dg)(af + bh) \\ &= aecf + aedh + bgcf + bgdh - cea f - cebh - dga f - dgbh \\ &= adeh + bcfg - bceh - adfg = (ad - bc)(eh - fg) = \det(\mathbf{A}) \det(\mathbf{B}). \end{aligned}$$

This remarkable fact is proved for  $n \times n$  matrices in Linear Algebra courses.

**Remark.** Notice that if  $\det(\mathbf{AB}) \neq 0$  then the above fact implies that both  $\det(\mathbf{A}) \neq 0$  and  $\det(\mathbf{B}) \neq 0$ . This is consistent with the fact given earlier that if  $\mathbf{AB}$  is invertible then both  $\mathbf{A}$  and  $\mathbf{B}$  are invertible

Other important facts about determinants include the following.

**Fact.** If  $\mathbf{A}$  is an  $n \times n$  matrix then

$$\det(\overline{\mathbf{A}}) = \overline{\det(\mathbf{A})}, \quad \det(\mathbf{A}^T) = \det(\mathbf{A}), \quad \det(\mathbf{A}^*) = \overline{\det(\mathbf{A})}.$$

**Reason.** It is easy to prove these facts for  $2 \times 2$  matrices. These facts are proved for  $n \times n$  matrices in Linear Algebra courses.

**Remark.** These are consistent with the facts given earlier that if  $\mathbf{A}$  is invertible then so are  $\overline{\mathbf{A}}$ ,  $\mathbf{A}^T$ , and  $\mathbf{A}^*$ .