

Solutions to Homework Problems on the Green Function Method II
Fall 2012, Math 246, Professor David Levermore

1. Compute the Green function $G(t, s)$ for the differential operator $L(t)$ defined by

$$L(t)y = D^2y - 2tDy + (t^2 - 1)y,$$

given that $e^{\frac{1}{2}t^2}$ and $te^{\frac{1}{2}t^2}$ solve the homogeneous equation $L(t)y = 0$. Use the result to solve the initial-value problem

$$y'' - 2ty' + (t^2 - 1)y = t^2e^{\frac{1}{2}t^2}, \quad y(0) = y'(0) = 0.$$

Solution. The Green function is given by

$$G(t, s) = \frac{\det \begin{pmatrix} e^{\frac{1}{2}s^2} & se^{\frac{1}{2}s^2} \\ e^{\frac{1}{2}t^2} & te^{\frac{1}{2}t^2} \end{pmatrix}}{\det \begin{pmatrix} e^{\frac{1}{2}s^2} & se^{\frac{1}{2}s^2} \\ se^{\frac{1}{2}s^2} & (s^2 + 1)e^{\frac{1}{2}s^2} \end{pmatrix}} = (t - s)e^{\frac{1}{2}t^2}e^{-\frac{1}{2}s^2}.$$

Because the equation is in normal form the forcing is $t^2e^{\frac{1}{2}t^2}$. The solution is therefore

$$y(t) = \int_0^t G(t, s) s^2 e^{\frac{1}{2}s^2} ds = e^{\frac{1}{2}t^2} \int_0^t (t - s)s^2 ds = \frac{1}{12}t^4 e^{\frac{1}{2}t^2}.$$

2. Compute the Green function $G(t, s)$ for the differential operator $L(t)$ defined by

$$L(t)y = tD^2y + (t - 1)Dy - y,$$

given that $t - 1$ and e^{-t} solve the homogeneous equation $L(t)y = 0$. Use the result to solve the initial-value problem

$$ty'' + (t - 1)y' - y = 2t^3, \quad y(1) = y'(1) = 0.$$

Solution. The Green function is given by

$$G(t, s) = \frac{\det \begin{pmatrix} s - 1 & e^{-s} \\ t - 1 & e^{-t} \end{pmatrix}}{\det \begin{pmatrix} s - 1 & e^{-s} \\ 1 & -e^{-s} \end{pmatrix}} = \frac{(t - 1)e^{-s} - e^{-t}(s - 1)}{se^{-s}}.$$

Because the normal form of the equation is

$$y'' + \left(1 - \frac{1}{t}\right)y' - \frac{1}{t}y = 2t^2,$$

the forcing is $2t^2$. The solution is therefore

$$y(t) = \int_1^t G(t, s) 2s^2 ds = 2 \int_1^t (t - 1)s - e^{-t}(s^2 - s)e^s ds = t^3 - 3t^2 + 5t - 5 + 2e^{1-t}.$$