

Final Exam Sample Problems, Math 246, Fall 2012

- (1) Consider the differential equation $\frac{dy}{dt} = (9 - y^2)y^2$.
- (a) Identify its equilibrium (stationary) points and classify their stability.
 - (b) Sketch how solutions move in the interval $-5 \leq y \leq 5$ (its phase-line portrait).
 - (c) If $y(0) = -1$, how does the solution $y(t)$ behave as $t \rightarrow \infty$?
- (2) Solve (possibly implicitly) each of the following initial-value problems. Identify their intervals of definition.

- (a) $\frac{dy}{dt} + \frac{2ty}{1+t^2} = t^2$, $y(0) = 1$.
- (b) $\frac{dy}{dx} + \frac{e^x y + 2x}{2y + e^x} = 0$, $y(0) = 0$.

- (3) Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)
t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = ti; y(1) = yi; h = (tf - ti)/n;
for i = 1:n
t(i + 1) = t(i) + h; y(i + 1) = y(i) + h*((t(i))^4 + (y(i))^2);
end
```

Suppose that the input values are $t_i = 1$, $y_i = 1$, $t_f = 5$, and $n = 40$.

- (a) What is the initial-value problem being approximated numerically?
 - (b) What is the numerical method being used?
 - (c) What is the timestep (step size)?
 - (d) What are the output values for the first two approximations, $y(2)$ and $y(3)$?
- (4) Give an explicit real-valued general solution of the following equations.
- (a) $y'' - 2y' + 5y = te^t + \cos(2t)$
 - (b) $u'' - 3u' - 10u = te^{-2t}$
 - (c) $v'' + 9v = \cos(3t)$

- (5) Solve the following initial-value problems.
- (a) $w'' + 4w' + 20w = 5e^{2t}$, $w(0) = 3$, $w'(0) = -7$.
 - (b) $y'' - 4y' + 4y = \frac{e^{2t}}{3+t}$, $y(0) = 0$, $y'(0) = 5$.
 - (c) $tu'' + 4u' = 0$, $u(1) = 2$, $u'(1) = -3$.

You should try to evaluate any definite integrals that arise.

- (6) Give an explicit general solution of the equation

$$h'' + 2h' + 5h = 0.$$

Sketch a typical solution for $t \geq 0$. If this equation governs a damped spring-mass system, is the system over, under, or critically damped?

- (7) When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.5 meters. (Gravitational acceleration is 9.8 m/sec^2 .) At $t = 0$ the mass is set in motion from 0.3 meters below its equilibrium (rest) position with a upward velocity of 2 m/sec. It is acted upon by an external force of $2 \cos(5t)$. Neglect drag and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem; just write it down!)

- (8) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial-value problem

$$y'' + 4y' + 8y = f(t), \quad y(0) = 2, \quad y'(0) = 4.$$

where

$$f(t) = \begin{cases} 4 & \text{for } 0 \leq t < 2, \\ t^2 & \text{for } 2 \leq t. \end{cases}$$

You may refer to the Laplace table in Section 6.2 of the book. (DO NOT take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

- (9) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by

$$(a) \quad Y(s) = \frac{e^{-3s}4}{s^2 - 6s + 5}, \quad (b) \quad Y(s) = \frac{e^{-2s}s}{s^2 + 4s + 8}.$$

You may refer to the Laplace table in Section 6.2 of the book.

- (10) Consider two interconnected tanks filled with brine (salt water). The first tank contains 80 liters and the second contains 30 liters. Brine flows with a concentration of 3 grams of salt per liter flows into the first tank at a rate of 2 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 6 liters per hour, from the second to the first at a rate of 4 liters per hour, and from the second into a drain at a rate of 2 liters per hour. At $t = 0$ there are 7 grams of salt in the first tank and 25 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

- (11) Consider the real vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3 + t^4 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Suppose that \mathbf{x}_1 and \mathbf{x}_2 comprise a fundamental set of solutions to the linear system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$. Give a general solution to this system.

- (12) Give a general real vector-valued solution of the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for

$$(a) \quad \mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}, \quad (b) \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

- (13) What answer will be produced by the following MATLAB command?

$$\gg \mathbf{A} = [1 \ 4; 3 \ 2]; [\text{vect}, \text{val}] = \text{eig}(\text{sym}(\mathbf{A}))$$

You do not have to give the answer in MATLAB format.

(14) A real 2×2 matrix \mathbf{A} has eigenvalues 2 and -1 with associated eigenvectors

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

- (a) Give a general solution to the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- (b) Compute $e^{t\mathbf{A}}$.
- (c) Sketch a phase-plane portrait for this system and identify its type. Classify the stability of the origin. (Carefully mark all sketched trajectories with arrows!)

(15) Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_I$ and describe how its solution behaves as $t \rightarrow \infty$ for the following \mathbf{A} and \mathbf{x}_I .

$$(a) \quad \mathbf{A} = \begin{pmatrix} 3 & 10 \\ -5 & -7 \end{pmatrix}, \quad \mathbf{x}_I = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

$$(b) \quad \mathbf{A} = \begin{pmatrix} 8 & -5 \\ 5 & -2 \end{pmatrix}, \quad \mathbf{x}_I = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

(16) Consider the nonlinear planar system

$$\begin{aligned} x' &= 2xy, \\ y' &= 9 - 9x - y^2. \end{aligned}$$

- (a) Find all of its equilibrium (critical, stationary) points.
- (b) Find a nonconstant function $H(x, y)$ such that every trajectory of the system satisfies $H(x, y) = c$ for some constant c .
- (c) Classify the type and stability of each equilibrium (critical, stationary) point.
- (d) Sketch the level sets (contour lines) $H(x, y) = c$ for values of c corresponding to each saddle point. Use arrows to indicate the direction of the trajectory along each curve that you sketch!

(17) Consider the nonlinear planar system

$$\begin{aligned} x' &= -5y, \\ y' &= x - 4y - x^2. \end{aligned}$$

- (a) Find all of its equilibrium (critical, stationary) points.
- (b) Compute the Jacobian matrix (derivative matrix, linearization coefficient matrix) at each equilibrium (critical, stationary) point.
- (c) Classify the type and stability of each equilibrium (critical, stationary) point.
- (d) Sketch a plausible global phase-plane portrait. Use arrows to indicate the direction of the trajectory along each curve that you sketch!

(18) Consider the nonlinear planar system

$$\begin{aligned} x' &= x(3 - 3x + 2y), \\ y' &= y(6 - x - y). \end{aligned}$$

Do parts (a-d) as for the previous problem.

- (e) Why do solutions that start in the first quadrant stay in the first quadrant?