Solutions of Sample Problems for Third In-Class Exam
Math 246, Fall 2012, Professor David Levermore

(1) Compute the Laplace transform of $f(t) = t\,e^{3t}$ from its definition.

Solution. The definition of the Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} t\, e^{3t}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T t\, e^{(3-s)t}\,\mathrm{d}t\,.$$

This limit diverges to $+\infty$ for $s \leq 3$ because in that case

$$\int_0^T t\, e^{(3-s)t}\,\mathrm{d}t \geq \int_0^T t\,\mathrm{d}t = \frac{T^2}{2}\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

For $s > 3$ an integration by parts shows that

$$\begin{aligned}\int_0^T t\, e^{(3-s)t}\,\mathrm{d}t &= t\,\frac{e^{(3-s)t}}{3-s}\bigg|_0^T - \int_0^T \frac{e^{(3-s)t}}{3-s}\,\mathrm{d}t \\ &= \left(t\,\frac{e^{(3-s)t}}{3-s} - \frac{e^{(3-s)t}}{(3-s)^2} \right)\bigg|_0^T \\ &= \left(T\,\frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2} \right) + \frac{1}{(3-s)^2}\,.\end{aligned}$$

Hence, for $s > 3$ one has that

$$\begin{aligned}\mathcal{L}[f](s) &= \lim_{T\to\infty} \left[\left(T\,\frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2} \right) + \frac{1}{(3-s)^2} \right] \\ &= \frac{1}{(3-s)^2} + \lim_{T\to\infty} \left(T\,\frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2} \right) \\ &= \frac{1}{(3-s)^2}\,.\end{aligned}$$

(2) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \leq t < 2\pi\,, \\ t - 2\pi & \text{for } t \geq 2\pi\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 13\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where
$$\mathcal{L}[y](s) = Y(s)\,,$$
$$\mathcal{L}[y'](s) = sY(s) - y(0) = sY(s) - 4\,,$$
$$\mathcal{L}[y''](s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 4s - 1\,.$$
To compute $\mathcal{L}[f](s)$, first write f as
$$\begin{aligned} f(t) &= \big(1 - u(t-2\pi)\big)\cos(t) + u(t-2\pi)(t-2\pi) \\ &= \cos(t) - u(t-2\pi)\cos(t) + u(t-2\pi)(t-2\pi) \\ &= \cos(t) - u(t-2\pi)\cos(t-2\pi) + u(t-2\pi)(t-2\pi)\,. \end{aligned}$$
Referring to the table on the last page, item 6 with $c = 2\pi$ and $j(t) = \cos(t)$, item 6 with $c = 2\pi$ and $j(t) = t$, item 2 with $a = 0$ and $b = 1$, and item 1 with $n = 1$ and $a = 1$ then show that
$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}[\cos(t)](s) - \mathcal{L}[u(t-2\pi)\cos(t-2\pi)](s) + \mathcal{L}[u(t-2\pi)(t-2\pi)](s) \\ &= \mathcal{L}[\cos(t)](s) - e^{-2\pi s}\mathcal{L}[\cos(t)](s) + e^{-2\pi s}\mathcal{L}[t](s) \\ &= \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,. \end{aligned}$$
The Laplace transform of the initial-value problem then becomes
$$\big(s^2Y(s) - 4s - 1\big) + 4\big(sY(s) - 4\big) + 13Y(s) = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,,$$
which becomes
$$(s^2 + 4s + 13)Y(s) - 4s - 1 - 16 = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,.$$
Hence, $Y(s)$ is given by
$$Y(s) = \frac{1}{s^2 + 4s + 13}\left(4s + 17 + \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\right).$$

(3) Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a) $F(s) = \dfrac{2}{(s+5)^2}\,,$

Solution. Referring to the table on the last page, item 1 with $n = 1$ and $a = -5$ gives
$$\mathcal{L}[t\,e^{-5t}](s) = \frac{1}{(s+5)^2}\,.$$
Multiplying this by 2 yields
$$\mathcal{L}[2t\,e^{-5t}](s) = \frac{2}{(s+5)^2}\,.$$
You therefore conclude that
$$\mathcal{L}^{-1}\left[\frac{2}{(s+5)^2}\right](t) = 2t\,e^{-5t}\,.$$

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}$,

Solution. The denominator factors as $(s-3)(s+2)$, so the partial fraction decomposition is

$$\frac{3s}{s^2 - s - 6} = \frac{3s}{(s-3)(s+2)} = \frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}.$$

Referring to the table on the last page, item 1 with $n = 0$ and $a = 3$, and with $n = 0$ and $a = -2$ gives

$$\mathcal{L}[e^{3t}](s) = \frac{1}{s-3}, \qquad \mathcal{L}[e^{-2t}](s) = \frac{1}{s+2},$$

whereby

$$\frac{3s}{s^2 - s - 6} = \tfrac{9}{5}\mathcal{L}[e^{3t}](s) + \tfrac{6}{5}\mathcal{L}[e^{-2t}](s) = \mathcal{L}\left[\tfrac{9}{5}e^{3t} + \tfrac{6}{5}e^{-2t}\right](s).$$

You therefore conclude that

$$\mathcal{L}^{-1}\left[\frac{3s}{s^2 - s - 6}\right](t) = \tfrac{9}{5}e^{3t} + \tfrac{6}{5}e^{-2t}.$$

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}$.

Solution. Complete the square in the denominator to get $(s-2)^2+1$. Referring to the table on the last page, item 2 with $a = 2$ and $b = 1$ gives

$$\mathcal{L}[e^{2t}\cos(t)](s) = \frac{s-2}{(s-2)^2 + 1}.$$

Item 6 with $c = 3$ and $j(t) = e^{2t}\cos(t)$ then gives

$$\mathcal{L}[u(t-3)e^{2(t-3)}\cos(t-3)](s) = e^{-3s}\frac{s-2}{(s-2)^2 + 1}.$$

You therefore conclude that

$$\mathcal{L}^{-1}\left[e^{-3s}\frac{s-2}{s^2 - 4s + 5}\right](t) = u(t-3)e^{2(t-3)}\cos(t-3).$$

(4) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

(a) $\mathbf{A}^T$,

Solution. The transpose of $\mathbf{A}$ is

$$\mathbf{A}^T = \begin{pmatrix} -i2 & 2+i \\ 1+i & -4 \end{pmatrix}.$$

(b) $\overline{\mathbf{A}}$,

Solution. The conjugate of $\mathbf{A}$ is

$$\overline{\mathbf{A}} = \begin{pmatrix} i2 & 1-i \\ 2-i & -4 \end{pmatrix} .$$

(c) $\mathbf{A}^*$,

Solution. The Hermitian transpose of $\mathbf{A}$ is

$$\mathbf{A}^* = \begin{pmatrix} i2 & 2-i \\ 1-i & -4 \end{pmatrix} .$$

(d) $5\mathbf{A} - \mathbf{B}$,

Solution. The difference of $5\mathbf{A}$ and $\mathbf{B}$ is given by

$$5\mathbf{A} - \mathbf{B} = \begin{pmatrix} -i10 & 5+i5 \\ 10+i5 & -20 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} -7-i10 & -1+i5 \\ 2+i5 & -27 \end{pmatrix} .$$

(e) $\mathbf{AB}$,

Solution. The product of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -i2\cdot 7 + (1+i)\cdot 8 & -i2\cdot 6 + (1+i)\cdot 7 \\ (2+i)\cdot 7 - 4\cdot 8 & (2+i)\cdot 6 - 4\cdot 7 \end{pmatrix} \\ &= \begin{pmatrix} 8-i6 & 7-i5 \\ -18+i7 & -16+i6 \end{pmatrix} . \end{aligned}$$

(f) $\mathbf{B}^{-1}$.

Solution. Observe that it is clear that $\mathbf{B}$ has an inverse because

$$\det(\mathbf{B}) = \det\begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = 7\cdot 7 - 6\cdot 8 = 49 - 48 = 1 .$$

The inverse of $\mathbf{B}$ is given by

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} .$$

(5) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix} .$$

(a) Find all the eigenvalues of $\mathbf{A}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 15 = (z-1)^2 - 16 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 1 ± 4, or simply -3 and 5.

(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.

Solution (using the Cayley-Hamilton method from notes). One has

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix} , \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix} .$$

Every nonzero column of $\mathbf{A} - 5\mathbf{I}$ has the form

$$\alpha_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0 .$$

These are all the eigenvectors associated with -3. Similarly, every nonzero column of $\mathbf{A} + 3\mathbf{I}$ has the form

$$\alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0 .$$

These are all the eigenvectors associated with 5.

(c) Diagonalize $\mathbf{A}$.

Solution. If you use the eigenpairs

$$\left(-3 , \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) , \qquad \left(5 , \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right) ,$$

then set

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} , \qquad \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} .$$

Because $\det(\mathbf{V}) = 1 \cdot 2 - (-2) \cdot 3 = 2 + 6 = 8$, you see that

$$\mathbf{V}^{-1} = \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} .$$

You conclude that $\mathbf{A}$ has the diagonalization

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} .$$

You do not have to multiply these matrices out. Had you started with different eigenpairs, the steps would be the same as above but you would obtain a different diagonalization.

(6) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix} ,$$

find all the eigenvectors of $\mathbf{A}$ associated with 1.

Solution. The eigenvectors of $\mathbf{A}$ associated with 1 are all nonzero vectors $\mathbf{v}$ that satisfy $\mathbf{A}\mathbf{v} = \mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ that satisfy $(\mathbf{A}-\mathbf{I})\mathbf{v} = 0$, which is

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 .$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} v_1 - v_2 + v_3 &= 0 , \\ v_1 \phantom{{}-v_2} - v_3 &= 0 , \\ - v_2 + 2v_3 &= 0 . \end{aligned}$$

You may solve this system either by elimination or by row reduction. By either method you find that its general solution is

$$v_1 = \alpha , \quad v_2 = 2\alpha , \quad v_3 = \alpha , \qquad \text{for any constant } \alpha .$$

The eigenvectors of $\mathbf{A}$ associated with 1 therefore have the form

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \text{for any nonzero constant } \alpha .$$

(7) Transform the equation $\dfrac{\mathrm{d}^3 u}{\mathrm{d}t^3} + t^2 \dfrac{\mathrm{d}u}{\mathrm{d}t} - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

Solution. Because the equation is third order, the first order system must have dimension three. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \sinh(2t) + 3x_1 - t^2 x_2 \end{pmatrix} , \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix} .$$

(8) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. The rates work out so there will always be 100 liters of brine in the first tank and 50 liters in the second. Let $S_1(t)$ be the grams of salt in the first tank and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\begin{aligned} \frac{dS_1}{dt} &= 2\cdot 3 + \frac{S_2}{50}2 - \frac{S_1}{100}5\,, \qquad && S_1(0) = 5\,,\\ \frac{dS_2}{dt} &= \frac{S_1}{100}5 - \frac{S_2}{50}2 - \frac{S_2}{50}3\,, \qquad && S_2(0) = 20\,. \end{aligned}$$

You could leave the answer in the above form. It can however be simplified to

$$\begin{aligned} \frac{dS_1}{dt} &= 6 + \frac{S_2}{25} - \frac{S_1}{20}\,, \qquad && S_1(0) = 5\,,\\ \frac{dS_2}{dt} &= \frac{S_1}{20} - \frac{S_2}{10}\,, \qquad && S_2(0) = 20\,. \end{aligned}$$

(9) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} = 3t^4 + 9 - 2t^4 = t^4 + 9\,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}$. Because $\dfrac{\mathbf{\Psi}(t)}{dt} = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\begin{aligned} \mathbf{A}(t) &= \frac{\mathbf{\Psi}(t)}{dt}\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix}\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{t^4+9}\begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix}\begin{pmatrix} 3 & -t^2 \\ -2t^2 & t^4+3 \end{pmatrix} = \frac{1}{t^4+9}\begin{pmatrix} 8t^3 & 6t - 2t^5 \\ 12t & -4t^3 \end{pmatrix}. \end{aligned}$$

(c) Give a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b).

Solution. Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a fundamental matrix for the system found in part (b) is simply given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}.$$

(d) For the system found in part (b), solve the initial-value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}\,, \qquad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Solution. Because a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b) was given in part (c), the solution of the initial-value problem is

$$\begin{aligned}\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(1)^{-1}\mathbf{x}(1) &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \frac{1}{10} \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 3t^4+9-2t^2 \\ 6t^2-6 \end{pmatrix} .\end{aligned}$$

Alternative Solution. Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 3 \end{pmatrix} .$$

The initial condition then implies that

$$\mathbf{x}(1) = c_1 \begin{pmatrix} 4 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4c_1 + c_2 \\ 2c_1 + 3c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

from which we see that $c_1 = \frac{3}{10}$ and $c_2 = -\frac{1}{5}$. The solution of the initial-value problem is thereby

$$\mathbf{x}(t) = \tfrac{3}{10} \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix} - \tfrac{1}{5} \begin{pmatrix} t^2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}t^4 - \frac{1}{5}t^2 + \frac{9}{10} \\ \frac{3}{5}t^2 - \frac{3}{5} \end{pmatrix} .$$

(10) Compute $e^{t\mathbf{A}}$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z-1)^2 - 4 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 1 ± 2. One then has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cosh(2t)\mathbf{I} + \frac{\sinh(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[\cosh(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(2t)}{2} \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix} . \end{aligned}$$

Alternative Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z+1)(z-3) .$$

The associated second-order general initial-value problem is

$$y'' - 2y' - 3y = 0 , \qquad y(0) = y_0 , \quad y'(0) = y_1 .$$

This has the general solution $y(t) = c_1 e^{3t} + c_2 e^{-t}$. Because $y'(t) = 3c_1 e^{3t} - c_2 e^{-t}$, the general initial conditions yield

$$y_0 = y(0) = c_1 + c_2\,, \qquad y_1 = y'(0) = 3c_1 - c_2\,.$$

This system can be solved to obtain

$$c_1 = \frac{y_0 + y_1}{4}\,, \qquad c_2 = \frac{3y_0 - y_1}{4}\,.$$

The solution of the general initial-value problem is thereby

$$y(t) = \frac{y_0 + y_1}{4}\, e^{3t} + \frac{3y_0 - y_1}{4}\, e^{-t} = \frac{e^{3t} + 3e^{-t}}{4}\, y_0 + \frac{e^{3t} - e^{-t}}{4}\, y_1\,.$$

The associated natural fundamental set of solutions is therefore

$$N_0(t) = \frac{e^{3t} + 3e^{-t}}{4}\,, \qquad N_1(t) = \frac{e^{3t} - e^{-t}}{4}\,,$$

whereby

$$\begin{aligned} e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A} &= \frac{e^{3t} + 3e^{-t}}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{3t} - e^{-t}}{4} \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{3t} + 2e^{-t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}\,. \end{aligned}$$

Second Alternative Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z+1)(z-3)\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -1 and 3. Because

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\,, \qquad \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\,,$$

you can read off that $\mathbf{A}$ has the eigenpairs

$$\left(-1\,, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right)\,, \qquad \left(3\,, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)\,.$$

Set

$$\mathbf{V} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}\,, \qquad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\,.$$

Because $\det(\mathbf{V}) = 4$, you see that

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{V} e^{t\mathbf{D}} \mathbf{V}^{-1} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & -2e^{-t} \\ e^{3t} & 2e^{3t} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{-t} + 2e^{3t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{-t} + 2e^{3t} \end{pmatrix}\,. \end{aligned}$$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z-4)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is 4, a double root. One then has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{4t}\Big[\mathbf{I} + t\left(\mathbf{A} - 4\mathbf{I}\right)\Big] \\ &= e^{4t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}\right] \\ &= e^{4t}\begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix} . \end{aligned}$$

Alternative Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z-4)^4 .$$

The associated second-order general initial-value problem is

$$y'' - 8y' + 16y = 0 , \qquad y(0) = y_0 , \quad y'(0) = y_1 .$$

This has the general solution $y(t) = c_1 e^{4t} + c_2 t e^{4t}$. Because $y'(t) = 4c_1 e^{3t} + 4c_2 t e^{-t} + c_2 e^{4t}$, the general initial conditions yield

$$y_0 = y(0) = c_1 , \qquad y_1 = y'(0) = 4c_1 + c_2 .$$

This system can be solved to obtain $c_1 = y_0$ and $c_2 = y_1 - 4y_0$. The solution of the general initial-value problem is thereby

$$y(t) = y_0\, e^{4t} + (y_1 - 4y_0)t\, e^{4t} = (1-4t)e^{4t}\, y_0 + t\, e^{4t}\, y_1 .$$

The associated natural fundamental set of solutions is therefore

$$N_0(t) = (1-4t)e^{4t} , \qquad N_1(t) = t\, e^{4t} ,$$

whereby

$$\begin{aligned} e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A} &= (1-4t)e^{4t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\, e^{4t}\begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix} \\ &= e^{4t}\begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix} . \end{aligned}$$

(11) Solve each of the following initial-value problems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} , \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - z - 12 = (z+3)(z-4) .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -3 and 4. These have the form $\frac{1}{2} \pm \frac{7}{2}$. One therefore has

$$
\begin{aligned}
e^{t\mathbf{A}} &= e^{\frac{1}{2}t}\left[\cosh\left(\tfrac{7}{2}t\right)\mathbf{I} + \frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}}\left(\mathbf{A} - \tfrac{1}{2}\mathbf{I}\right)\right] \\
&= e^{\frac{1}{2}t}\left[\cosh\left(\tfrac{7}{2}t\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}}\begin{pmatrix} \frac{3}{2} & 2 \\ 5 & -\frac{3}{2} \end{pmatrix}\right] \\
&= e^{\frac{1}{2}t}\begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}.
\end{aligned}
$$

The solution of the initial-value problem is therefore

$$
\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= e^{\frac{1}{2}t}\begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= e^{\frac{1}{2}t}\begin{pmatrix} \cosh\left(\frac{7}{2}t\right) - \frac{1}{7}\sinh\left(\frac{7}{2}t\right) \\ -\cosh\left(\frac{7}{2}t\right) + \frac{13}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}.
\end{aligned}
$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$ is given by

$$
p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 5 = (z-1)^2 + 4\,.
$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. One therefore has

$$
\begin{aligned}
e^{t\mathbf{A}} &= e^{t}\left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A} - \mathbf{I})\right] \\
&= e^{t}\left[\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2}\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\right] \\
&= e^{t}\begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix}.
\end{aligned}
$$

The solution of the initial-value problem is therefore

$$
\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= e^{t}\begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= e^{t}\begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) \\ -2\sin(2t) + \cos(2t) \end{pmatrix}.
\end{aligned}
$$

Remark. You could have used other methods to compute $e^{t\mathbf{A}}$ in each part of the above problem. Alternatively, you could have constructed a fundamental matrix $\mathbf{\Psi}(t)$ and then determined $\mathbf{c}$ so that $\mathbf{\Psi}(t)\mathbf{c}$ satisfies the initial conditions.

(12) Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 1 = (z-1)^2\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is 1, a double root. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t\big[\mathbf{I} + t\,(\mathbf{A} - \mathbf{I})\big] = e^t\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\right] \\ &= e^t\begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix}\,. \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t\begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1e^t\begin{pmatrix} 1+2t \\ t \end{pmatrix} + c_2e^t\begin{pmatrix} -4t \\ 1-2t \end{pmatrix}\,. \end{aligned}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $\pm i4$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= \left[\cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}\mathbf{A}\right] = \left[\cos(4t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4}\begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\right] \\ &= \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}\,. \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ \sin(4t) \end{pmatrix} + c_2\begin{pmatrix} -\frac{5}{4}\sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}\,. \end{aligned}$$

Alternative Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $\pm i4$. Because

$$\mathbf{A} - i4\mathbf{I} = \begin{pmatrix} 2 - i4 & -5 \\ 4 & -2 - i4 \end{pmatrix}, \qquad \mathbf{A} + i4\mathbf{I} = \begin{pmatrix} 2 + i4 & -5 \\ 4 & -2 + i4 \end{pmatrix},$$

you can read off that $\mathbf{A}$ has the eigenpairs

$$\left(i4\,, \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \right), \qquad \left(-i4\,, \begin{pmatrix} 1 - i2 \\ 2 \end{pmatrix} \right).$$

The system therefore has the complex-valued solution

$$\begin{aligned} e^{i4t} \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} &= \big(\cos(4t) + i \sin(4t) \big) \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(4t) - 2\sin(4t) + i2\cos(4t) + i\sin(4t) \\ 2\cos(4t) + i2\sin(4t) \end{pmatrix}. \end{aligned}$$

By taking real and imaginary parts, you obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re}\left(e^{i4t} \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix}, \\ \mathbf{x}_2(t) &= \operatorname{Im}\left(e^{i4t} \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix} + c_2 \begin{pmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{pmatrix}.$$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z - 3)^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $3 \pm i4$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t} \left[\cos(4t)\mathbf{I} + \frac{\sin(4t)}{4} (\mathbf{A} - 3\mathbf{I}) \right] \\ &= e^{3t} \left[\cos(4t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4} \begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix} \right] \\ &= e^{3t} \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{3t} \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{3t} \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ -\frac{5}{4}\sin(4t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Alternative Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z-3)^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $3 \pm i4$. Because

$$\mathbf{A} - (3+i4)\mathbf{I} = \begin{pmatrix} 2-i4 & 4 \\ -5 & -2-i4 \end{pmatrix}, \quad \mathbf{A} - (3-i4)\mathbf{I} = \begin{pmatrix} 2+i4 & 4 \\ -5 & -2+i4 \end{pmatrix},$$

you can read off that $\mathbf{A}$ has the eigenpairs

$$\left(3+i4\,, \begin{pmatrix} -2 \\ 1-i2 \end{pmatrix}\right), \qquad \left(3-i4\,, \begin{pmatrix} -2 \\ 1+i2 \end{pmatrix}\right).$$

The system therefore has the complex-valued solution

$$\begin{aligned} e^{(3+i4)t}\begin{pmatrix} -2 \\ 1-i2 \end{pmatrix} &= e^{3t}\big(\cos(4t) + i\sin(4t)\big)\begin{pmatrix} -2 \\ 1-i2 \end{pmatrix} \\ &= e^{3t}\begin{pmatrix} -2\cos(4t) - i2\sin(4t) \\ \cos(4t) + 2\sin(4t) + i\sin(4t) - i2\cos(4t) \end{pmatrix}. \end{aligned}$$

By taking real and imaginary parts, you obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re}\left(e^{(3+i4)t}\begin{pmatrix} -2 \\ 1-i2 \end{pmatrix}\right) = e^{3t}\begin{pmatrix} -2\cos(4t) \\ \cos(4t) + 2\sin(4t) \end{pmatrix}, \\ \mathbf{x}_2(t) &= \operatorname{Im}\left(e^{(3+i4)t}\begin{pmatrix} -2 \\ 1+i2 \end{pmatrix}\right) = e^{3t}\begin{pmatrix} -2\sin(4t) \\ \sin(4t) - 2\cos(4t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t}\begin{pmatrix} -2\cos(4t) \\ \cos(4t) + 2\sin(4t) \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} -2\sin(4t) \\ \sin(4t) - 2\cos(4t) \end{pmatrix}.$$

(13) Sketch the phase-plane portrait for each of the systems in the previous problem. Indicate typical trajectories. For each portrait identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

(a) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z-1)^2$, one sees that $\mu = 1$ and $\delta = 0$. Because

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix},$$

we see that the eigenvectors associated with 1 have the form

$$\alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{for some } \alpha \neq 0\,.$$

Because $\mu = 1 > 0$, $\delta = 0$, and $a_{21} > 0$ the phase portrait is a *counterclockwise twist source*. The origin is thereby *unstable*. The phase portrait should show there is one trajectory that emerges from the origin on each side of the line $y = x/2$. Every other trajectory emerges from the origin with a counterclockwise twist.

(b) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = z^2 + 16$, one sees that $\mu = 0$ and $\delta = -16$. There are no real eigenpairs. Because $\mu = 0$, $\delta = -16 < 0$, and $a_{21} > 0$ the phase portrait is a *counterclockwise center*. The origin is thereby *stable*. The phase portrait should indicate a family of counterclockwise elliptical trajectories that go around the origin.

(c) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z-3)^2 + 16$, one sees that $\mu = 3$ and $\delta = -16$. There are no real eigenpairs. Because $\mu = 3$, $\delta = -16 < 0$, and $a_{21} < 0$ the phase portrait is a *clockwise spiral source*. The origin is thereby *unstable*. The phase portrait should indicate a family of clockwise spiral trajectories that emerge from the origin.

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2 + b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2 + b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs} J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)$$

and u is the unit step function.