

Sample Problems for Third In-Class Exam
Math 246, Fall 2012, Professor David Levermore

- (1) Compute the Laplace transform of $f(t) = t e^{3t}$ from its definition.
- (2) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = f(t), \quad y(0) = 4, \quad y'(0) = 1,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \leq t < 2\pi, \\ t - 2\pi & \text{for } t \geq 2\pi. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

- (3) Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a) $F(s) = \frac{2}{(s+5)^2},$

(b) $F(s) = \frac{3s}{s^2 - s - 6},$

(c) $F(s) = \frac{(s-2)e^{-3s}}{s^2 - 4s + 5}.$

- (4) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

(a) $\mathbf{A}^T,$

(b) $\overline{\mathbf{A}},$

(c) $\mathbf{A}^*,$

(d) $5\mathbf{A} - \mathbf{B},$

(e) $\mathbf{AB},$

(f) $\mathbf{B}^{-1}.$

- (5) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}.$$

(a) Find all the eigenvalues of \mathbf{A} .

(b) For each eigenvalue of \mathbf{A} find all of its eigenvectors.

(c) Diagonalize \mathbf{A} .

- (6) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of \mathbf{A} associated with 1.

- (7) Transform the equation $\frac{d^3u}{dt^3} + t^2\frac{du}{dt} - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

- (8) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

- (9) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1, \mathbf{x}_2$ is a fundamental set of solutions to the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x},$$

wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) Give a fundamental matrix $\Psi(t)$ for the system found in part (b).

(d) For the system found in part (b), solve the initial-value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

- (10) Compute $e^{t\mathbf{A}}$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

- (11) Solve each of the following initial-value problems.

(a) $\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

(b) $\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

(12) Find a general solution for each of the following systems.

$$(a) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(b) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(c) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(13) Sketch the phase-plane portrait for each of the systems in the previous problem. Indicate typical trajectories. For each portrait identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \quad \text{for } s > a.$$

$$\mathcal{L}[e^{at} \cos(bt)](s) = \frac{s-a}{(s-a)^2 + b^2} \quad \text{for } s > a.$$

$$\mathcal{L}[e^{at} \sin(bt)](s) = \frac{b}{(s-a)^2 + b^2} \quad \text{for } s > a.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \quad \text{where } J(s) = \mathcal{L}[j(t)](s).$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \quad \text{where } J(s) = \mathcal{L}[j(t)](s).$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs} J(s) \quad \text{where } J(s) = \mathcal{L}[j(t)](s) \\ \text{and } u \text{ is the unit step function.}$$