Third In-Class Exam Solutions
Math 246, Professor David Levermore
Thursday, 29 November 2012

(1) [6] Given that 2 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 2.

Solution. The eigenvectors of $\mathbf{A}$ associated with 2 are all nonzero vectors $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = 2\mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ such that $(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$, which is

$$\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} 2v_2 - v_3 &= 0\,, \\ v_1 - v_2 \qquad &= 0\,, \\ 3v_1 - v_2 - v_3 &= 0\,. \end{aligned}$$

We may solve this system either by elimination or by row reduction. By either method we find that its general solution is

$$v_1 = \alpha\,, \quad v_2 = \alpha\,, \quad v_3 = 2\alpha\,, \qquad \text{for any constant } \alpha\,.$$

Therefore the eigenvectors of $\mathbf{A}$ associated with 2 have the form

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad \text{for any constant } \alpha \neq 0\,.$$

(2) [8] A 3×3 matrix $\mathbf{A}$ has the eigenpairs

$$\left(-3\,, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \qquad \left(2\,, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right), \qquad \left(5\,, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right).$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{A}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution (a). One choice for $\mathbf{V}$ and $\mathbf{D}$ is

$$\mathbf{V} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Solution (b). Use the given eigenpairs to construct the special solutions

$$\mathbf{x}_1(t) = e^{-3t}\begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \mathbf{x}_2(t) = e^{2t}\begin{pmatrix}-1\\1\\1\end{pmatrix}, \quad \mathbf{x}_3(t) = e^{5t}\begin{pmatrix}1\\-1\\2\end{pmatrix},$$

Then a fundamental matrix for the system is

$$\boldsymbol{\Psi}(t) = \begin{pmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t)\end{pmatrix} = \begin{pmatrix} e^{-3t} & -e^{2t} & e^{5t} \\ e^{-3t} & e^{2t} & -e^{5t} \\ 0 & e^{2t} & 2e^{5t}\end{pmatrix}.$$

Alternative Solution (b). Given the $\mathbf{V}$ and $\mathbf{D}$ from part (a), a fundamental matrix for the system is

$$\boldsymbol{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}} = \begin{pmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}e^{-3t} & 0 & 0\\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{5t}\end{pmatrix} = \begin{pmatrix} e^{-3t} & -e^{2t} & e^{5t} \\ e^{-3t} & e^{2t} & -e^{5t} \\ 0 & e^{2t} & 2e^{5t}\end{pmatrix}.$$

(3) [6] Recast the equation $u'''' = e^u u''' + (u')^2 + \sin(t)$ as a first-order system of ordinary differential equations.

Solution. Because the equation is fourth order, the first-order system must have dimension four. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}x_2\\x_3\\x_4\\ e^{x_1}x_4 + (x_2)^2 + \sin(t)\end{pmatrix}, \quad \text{where} \quad \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}u\\u'\\u''\\u'''\end{pmatrix}.$$

(4) [12] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix}1\\t^2\end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix}t^4\\t^6+4\end{pmatrix}$.

(a) [2] Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [4] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system you found in part (b).

(d) [4] Find the natural fundamental matrix associated with the initial time 0 for the system you found in part (b).

Solution (a). The Wronskian is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix}1 & t^4\\ t^2 & t^6+4\end{pmatrix} = 1\cdot(t^6+4) - t^2\cdot t^4 = t^6 + 4 - t^6 = 4\,.$$

Solution (b). Let $\boldsymbol{\Psi}(t) = \begin{pmatrix}1 & t^4\\ t^2 & t^6+4\end{pmatrix}$. Because $\boldsymbol{\Psi}'(t) = \mathbf{A}(t)\boldsymbol{\Psi}(t)$, we have

$$\begin{aligned}\mathbf{A}(t) = \boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(t)^{-1} &= \begin{pmatrix}0 & 4t^3\\ 2t & 6t^5\end{pmatrix}\begin{pmatrix}1 & t^4\\ t^2 & t^6+4\end{pmatrix}^{-1} \\ &= \frac{1}{4} = \begin{pmatrix}0 & 4t^3\\ 2t & 6t^5\end{pmatrix}\begin{pmatrix}t^6+4 & -t^4\\ -t^2 & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}-4t^5 & 4t^3\\ 8t-4t^7 & 4t^5\end{pmatrix} = \begin{pmatrix}-t^5 & t^3\\ 2t-t^7 & t^5\end{pmatrix}.\end{aligned}$$

Solution (c). A general solution is

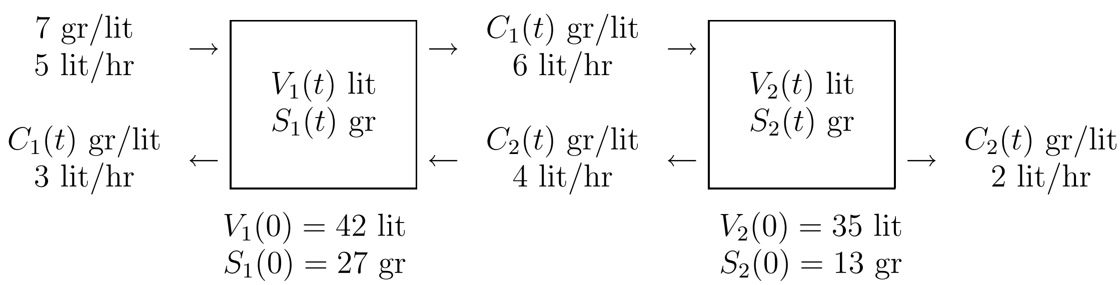
$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} 1 \\ t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^4 \\ t^6 + 4 \end{pmatrix} .$$

Solution (d). By using the fundamental matrix $\mathbf{\Psi}(t)$ from part (b) we find that the natural fundamental matrix associated with the initial time 0 is

$$\begin{aligned} \mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} &= \begin{pmatrix} 1 & t^4 \\ t^2 & t^6 + 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} 1 & t^4 \\ t^2 & t^6 + 4 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 4 & t^4 \\ 4t^2 & t^6 + 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{4}t^4 \\ t^2 & \frac{1}{4}t^6 + 1 \end{pmatrix} . \end{aligned}$$

(5) [6] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 42 liters and the second contains 35 liters. Brine with a salt concentration of 7 grams per liter flows into the first tank at 5 liters per hour. Well-stirred brine flows from the first tank into the second at 6 liters per hour, from the second into the first at 4 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 2 liters per hour. At $t = 0$ there are 27 grams of salt in the first tank and 13 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t minutes. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t minutes. Because mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.



We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will always be $V_1(t) = 42$ liters of brine in the first tank and $V_2(t) = 35$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 7 \cdot 5 + \frac{S_2}{35}4 - \frac{S_1}{42}6 - \frac{S_1}{42}3 , & S_1(0) &= 27 , \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{42}6 - \frac{S_2}{35}4 - \frac{S_2}{35}2 , & S_2(0) &= 13 . \end{aligned}$$

You could leave the answer in the above form. However, it can be simplified to

$$\begin{aligned}\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 35 + \tfrac{4}{35}S_2 - \tfrac{9}{42}S_1\,, \qquad & S_1(0) = 27\,,\\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \tfrac{1}{7}S_1 - \tfrac{6}{35}S_5\,, \qquad & S_2(0) = 13\,.\end{aligned}$$

(6) [16] Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 3z - 10 = (z+2)(z-5)\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -2 and 5. Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix},$$

we can read off that eigenpairs are

$$\left(-2\,, \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right), \qquad \left(5\,, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

Therefore a general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{pmatrix} -1 \\ 3 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Alternative Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 3z - 10 = (z+2)(z-5)\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -2 and 5. The average of these eigenvalues is $\frac{3}{2}$, so they can be expressed as $\frac{3}{2} \pm \frac{7}{2}$. Because

$$\begin{aligned} e^{t\mathbf{A}} &= e^{\frac{3}{2}t}\left[\cosh(\tfrac{7}{2}t)\mathbf{I} + \frac{\sinh(\frac{7}{2}t)}{\frac{7}{2}}(\mathbf{A} - \tfrac{3}{2}\mathbf{I})\right] \\ &= e^{\frac{3}{2}t}\left[\cosh(\tfrac{7}{2}t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(\frac{7}{2}t)}{\frac{7}{2}}\begin{pmatrix} \frac{5}{2} & 2 \\ 3 & -\frac{5}{2} \end{pmatrix}\right] \\ &= e^{\frac{3}{2}t}\begin{pmatrix} \cosh(\frac{7}{2}t) + \frac{5}{7}\sinh(\frac{7}{2}t) & \frac{4}{7}\sinh(\frac{7}{2}t) \\ \frac{6}{7}\sinh(\frac{7}{2}t) & \cosh(\frac{7}{2}t) - \frac{5}{7}\sinh(\frac{7}{2}t) \end{pmatrix}, \end{aligned}$$

a general solution is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} = c_1 e^{\frac{3}{2}t}\begin{pmatrix} \cosh(\frac{7}{2}t) + \frac{5}{7}\sinh(\frac{7}{2}t) \\ \frac{6}{7}\sinh(\frac{7}{2}t) \end{pmatrix} + c_2 e^{\frac{3}{2}t}\begin{pmatrix} \frac{4}{7}\sinh(\frac{7}{2}t) \\ \cosh(\frac{7}{2}t) - \frac{5}{7}\sinh(\frac{7}{2}t) \end{pmatrix}.$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 & 5 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -7 & 5 \\ -5 & 3 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 4 = (z+2)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is only -2. Because

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-2t}\left[\mathbf{I} + t\left(\mathbf{A} + 2\mathbf{I}\right)\right] \\ &= e^{-2t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\right] = e^{-2t}\begin{pmatrix} 1-5t & 5t \\ -5t & 1+5t \end{pmatrix} , \end{aligned}$$

a general solution is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} = c_1 e^{-2t}\begin{pmatrix} 1-5t \\ -5t \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 5t \\ 1+5t \end{pmatrix} .$$

Alternative Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -7 & 5 \\ -5 & 3 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 4 = (z+2)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is only -2. Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix} ,$$

we can read off that one eigenpair is

$$\left(-2\,,\ \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) .$$

We can use this eigenpair to construct the solution

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

A second solution can be constructed by the formula

$$\mathbf{x}_2(t) = e^{-2t}\mathbf{w} + t\,e^{-2t}(\mathbf{A} + 2\mathbf{I})\mathbf{w} ,$$

where $\mathbf{w}$ is any nonzero vector that is *not* an eigenvector for the eigenvalue -2. For example, we can set

$$\begin{aligned} \mathbf{x}_2(t) &= e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\,e^{-2t}\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\,e^{-2t}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = e^{3t}\begin{pmatrix} 5t \\ 1+5t \end{pmatrix} . \end{aligned}$$

In that case a general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 5t \\ 1+5t \end{pmatrix} .$$

(7) [12] Sketch phase-plane portraits for *both* of the systems in the previous problem. For each portrait identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

Solution (a). Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z+2)(z-5)$, its eigenvalues are -2 and 5. Because these are real with opposite signs, the phase portrait is a *saddle*. The origin is thereby *unstable*, but not repelling. Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix},$$

we can read off that eigenpairs are

$$\left(-2\,, \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right), \qquad \left(5\,, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

The solutions associated with the eigenvalue -2 will lie on the line $y = -3x$ and move toward the origin. The solutions associated with the eigenvalue 5 will lie on the line $y = \frac{1}{2}x$ and move away from the origin. Every other trajectory will sweep in moving away from the line $y = -3x$ and sweep out moving toward the line $y = \frac{1}{2}x$.

Solution (b). Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z+2)^2$, one sees that $\mu = -2$ and $\delta = 0$. Because $\mu = -2 < 0$, $\delta = 0$, and $a_{21} = -5 < 0$ the phase portrait is a *clockwise twist sink*. The origin is thereby *attracting* (and also *stable*). Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix},$$

we see that the eigenvectors associated with -2 have the form

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for some } \alpha \neq 0\,.$$

The associated solutions will lie on the line $y = x$ and move towards the origin. Every other trajectory approaches the origin tangent to the line $y = x$ with a clockwise twist.

(8) [6] Given that $e^{t\mathbf{A}} = e^{2t}\begin{pmatrix} \cos(3t) & -3\sin(3t) \\ \frac{1}{3}\sin(3t) & \cos(3t) \end{pmatrix}$ for $\mathbf{A} = \begin{pmatrix} 2 & -9 \\ 1 & 2 \end{pmatrix}$, do the following.

(a) [2] Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}\,, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

(b) [4] Sketch a phase-plane portrait, identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

Solution (a). The solution of the initial-value problem is given by

$$\mathbf{x}(t) = e^{t\mathbf{A}}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = e^{2t}\begin{pmatrix} \cos(3t) & -3\sin(3t) \\ \frac{1}{3}\sin(3t) & \cos(3t) \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = e^{2t}\begin{pmatrix} 3\cos(3t) + 3\sin(3t) \\ \sin(3t) - \cos(3t) \end{pmatrix}.$$

Solution (b). We can read off from the form of $e^{t\mathbf{A}}$ that the eigenvalues of $\mathbf{A}$ are $2 \pm i3$. Because $\mu = 2 > 0$, $\delta = -9 < 0$, and $a_{21} = 1 > 0$ the phase portrait is a *counterclockwise spiral source*. The origin is thereby *repelling* (and also *unstable*). Every other trajectory moves away from the origin with a counterclockwise spiral.

(9) [8] Compute the Laplace transform of $f(t) = u(t-2)\,e^{-3t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} u(t-2)\,e^{-3t}\,\mathrm{d}t = \lim_{T\to\infty} \int_2^T e^{-(s+3)t}\,\mathrm{d}t\,.$$

When $s \le -3$ this limit diverges to $+\infty$ because in that case we have for every $T > 2$

$$\int_2^T e^{-(s+3)t}\,\mathrm{d}t \ge \int_2^T \mathrm{d}t = T - 2\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

When $s > -3$ we have for every $T > 2$

$$\int_2^T e^{-(s+3)t}\,\mathrm{d}t = -\frac{e^{-(s+3)t}}{s+3}\bigg|_2^T = -\frac{e^{-(s+3)T}}{s+3} + \frac{e^{-(s+3)2}}{s+3}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[-\frac{e^{-(s+3)T}}{s+3} + \frac{e^{-(s+3)2}}{s+3}\right] = \frac{e^{-(s+3)2}}{s+3}\,.$$

(10) [8] Find the inverse Laplace transforms of the function

$$F(s) = e^{-2s}\,\frac{s-1}{s^2 - 2s - 15}\,.$$

You may refer to the table on the last page.

Solution. The denominator factors as $(s-5)(s+3)$, so the partial fraction decomposition is

$$\frac{s-1}{s^2-2s-15} = \frac{s-1}{(s-5)(s+3)} = \frac{\frac{1}{2}}{s-5} + \frac{\frac{1}{2}}{s+3}\,.$$

Referring to the table on the last page, item 1 with $a = 5$ and $n = 0$, and with $a = -3$ and $n = 0$ shows that

$$\mathcal{L}[e^{5t}](s) = \frac{1}{s-5}\,, \qquad \mathcal{L}[e^{-3t}](s) = \frac{1}{s+3}\,.$$

These formulas also can be obtained from item 2 with $a = 5$ and $b = 0$, and with $a = -3$ and $b = 0$. From these formulas we obtain

$$\mathcal{L}^{-1}\left[\frac{s-1}{s^2-2s-15}\right](t) = \tfrac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-5}\right] + \tfrac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = \tfrac{1}{2}e^{5t} + \tfrac{1}{2}e^{-3t}\,.$$

Then item 6 with $c = 5$ and $j(t) = \frac{1}{2}e^{5t} + \frac{1}{2}e^{-3t}$ shows that

$$\begin{aligned}\mathcal{L}^{-1}\left[e^{-2s}\frac{s-1}{s^2-2s-15}\right](t) &= u(t-2)\,\mathcal{L}^{-1}\left[\frac{s-1}{s^2-2s-15}\right](t-2)\\ &= u(t-2)\left(\tfrac{1}{2}e^{5(t-2)} + \tfrac{1}{2}e^{-3(t-2)}\right).\end{aligned}$$

(11) [12] Consider the following MATLAB commands.

```
>> syms t s Y; f = ['heaviside(t)*t^2 + heaviside(t − 3)*(3*t − t^2)'];
>> diffeqn = sym('D(D(y))(t) − 6*D(y)(t) + 10*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s)', 'y(0)', 'D(y)(0)'}, {Y, 2, 3});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) Give the initial-value problem for $y(t)$ that is being solved.
(b) Find the Laplace transform $Y(s)$ of the solution $y(t)$.
You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution (a). The initial-value problem for $y(t)$ that is being solved is

$$y'' - 6y' + 10y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 3\,,$$

where the forcing $f(t)$ can be expressed either as

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 3\,, \\ 3t & \text{for } 3 \le t\,, \end{cases}$$

or in terms of the unit step function as $f(t) = t^2 + u(t-3)(3t - t^2)$.

Solution (b). The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) - 6\mathcal{L}[y'](s) + 10\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 2\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 3\,. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, we first write $f(t)$ as

$$f(t) = t^2 + u(t-3)(3t - t^2) = t^2 + u(t-3)j(t-3)\,,$$

where by setting $j(t-3) = 3t - t^2$ we see that

$$j(t) = 3(t+3) - (t+3)^2 = 3t + 9 - t^2 - 6t - 9 = -t^2 - 3t\,.$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 2$ and with $a = 0$ and $n = 1$ shows that

$$\mathcal{L}[t^2](s) = \frac{2}{s^3}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,,$$

whereby item 6 with $c = 3$ and $j(t) = -t^2 - 3t$ shows that

$$\mathcal{L}\big[u(t-3)j(t-3)\big](s) = e^{-3s}\mathcal{L}[j](s) = -e^{-3s}\mathcal{L}\big[t^2 + 3t\big](s) = -e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right)\,.$$

Therefore

$$\mathcal{L}[f](s) = \mathcal{L}\big[t^2 + u(t-3)j(t-3)\big](s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right)\,.$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 2s - 3\big) - 6\big(sY(s) - 2\big) + 10Y(s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right),$$

which becomes

$$(s^2 - 6s + 10)Y(s) - 2s + 9 = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 - 6s + 10}\left(2s - 9 + \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right)\right).$$

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2 + b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2 + b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)$$

and u is the unit step function.