Second In-Class Exam Solutions
Math 246, Professor David Levermore
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(1) [4] Give the interval of definition for the solution of the initial-value problem

$$z''' + \frac{7}{t^2 - 25}\, z' = \frac{3}{\sin(t)}\,, \qquad z(-4) = z'(-4) = z''(-4) = 0\,.$$

Solution. The equation is linear and is already in normal form. The coefficient is undefined at $t = \pm 5$ and is continuous everywhere else. The forcing is undefined at $t = n\pi$ for every integer n and is continuous everywhere else. Therefore the interval of definition is $(-5, -\pi)$ because

- the initial time $t = -4$ is in the interval $(-5, -\pi)$,
- the coefficient and forcing are both continuous over the interval $(-5, -\pi)$,
- the forcing is undefined at $-\pi$ while the coefficient is undefined at -5.

(2) [10] The functions e^{3t} and $t\,e^{3t}$ are a fundamental set of solutions to $y'' - 6y' + 9y = 0$.

(a) Find the solution $Y(t)$ to the general initial-value problem

$$y'' - 6y' + 9y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

(b) Find the associated natural fundamental set of solutions to $y'' - 6y' + 9y = 0$.

Solution (a). Since e^{3t} and $t\,e^{3t}$ is a fundamental set of solutions to $y'' - 6y' + 9y = 0$, a general solution is $y(t) = c_1 e^{3t} + c_2 t\,e^{3t}$. Because $y'(t) = 3c_1 e^{3t} + 3c_2 t\,e^{3t} + c_2 e^{3t}$, the initial conditions imply

$$y_0 = y(0) = c_1\,, \qquad y_1 = y'(0) = 3c_1 + c_2\,.$$

We solve these equations to obtain

$$c_1 = y_0\,, \qquad c_2 = y_1 - 3y_0\,.$$

Therefore the solution to the general initial-value problem is

$$y(t) = y_0 e^{3t} + (y_1 - 3y_0)t\,e^{3t}\,.$$

Solution (b). Because the above solution to the general initial-value problem can be written as

$$y(t) = (1 - 3t)e^{3t}\,y_0 + t\,e^{3t}\,y_1\,,$$

the associated natural fundamental set of solutions is seen to be

$$N_0(t) = (1 - 3t)e^{3t}\,, \qquad N_1(t) = t\,e^{3t}\,.$$

(3) [4] Suppose that $X_1(t)$, $X_2(t)$, and $X_3(t)$ are solutions of the differential equation

$$x''' - x'' + e^t x' - t^2 x = 0\,,$$

Suppose you know that $W[X_1, X_2, X_3](0) = 3$. What is $W[X_1, X_2, X_3](t)$?

Solution. Abel's Theorem states that $w(t) = W[X_1, X_2, X_3](t)$ satisfies $w' - w = 0$. It follows that $w(t) = w(0)e^t$. Because $w(0) = W[Z_1, Z_2, Z_3](0) = 3$, we obtain $w(t) = 3e^t$. Therefore

$$W[X_1, X_2, X_3](t) = 3e^t\,.$$

(4) [10] Give a general real solution of the equation

$$\mathrm{D}^2x - 4\mathrm{D}x - 21x = 12\cos(3t) - 7e^{4t}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a nonhomogeneous linear equation with constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 4z - 21 = (z+3)(z-7)\,,$$

which has the two simple real roots -3 and 7. Therefore a general solution of the associated homogeneous equation is

$$x_H(t) = c_1 e^{-3t} + c_2 e^{7t}\,.$$

The forcing term $12\cos(3t)$ has degree $d = 0$ and characteristic $\mu + i\nu = i3$, which is a root of $p(z)$ of multiplicity $m = 0$. The forcing term $-7e^{4t}$ has degree $d = 0$ and characteristic $\mu + i\nu = 4$, which is a root of $p(z)$ of multiplicity $m = 0$. Because the forcing has composite characteristic form, either Key Identity Evaluations or Undetermined Coefficients can be used to find particular solutions $x_{P1}(t)$ and $x_{P2}(t)$ that satisfy

$$\mathrm{L}(x_{P1}(t)) = 12\cos(3t)\,, \qquad \mathrm{L}(x_{P2}(t)) = -7e^{4t}\,.$$

Then a general solution will be given by $x(t) = x_H(t) + x_{P1}(t) + x_{P2}(t)$.

Key Indentity Evaluations. Because $m = 0$ and $m + d = 0$ for both forcing terms, we need only the Key Identity

$$\mathrm{L}\big(e^{zt}\big) = p(z)e^{zt} = (z^2 - 4z - 21)e^{zt}\,.$$

By evaluating this at $z = i3$ and $z = 4$ we obtain

$$\begin{aligned}
\mathrm{L}\big(e^{i3t}\big) &= \big((i3)^2 - 4(i3) - 21\big)e^{i3t} = (-9 - i12 - 21)e^{i3t} = -(30 + i12)e^{i3t}\,,\\
\mathrm{L}\big(e^{4t}\big) &= \big(4^2 - 4\cdot 4 - 21\big)e^{4t} = (16 - 16 - 21)e^{4t} = -21e^{4t}\,.
\end{aligned}$$

Because the first forcing term is $12\cos(3t) = 12\,\mathrm{Re}(e^{i3t})$, we write

$$\mathrm{L}\left(-\frac{12e^{i3t}}{30 + i12}\right) = 12e^{i3t}\,,$$

which implies that

$$\begin{aligned}
x_{P1}(t) &= -\,\mathrm{Re}\left(\frac{12e^{i3t}}{30 + i12}\right) = -2\,\mathrm{Re}\left(\frac{e^{i3t}}{5 + i2}\,\frac{5 - i2}{5 - i2}\right) = -2\,\mathrm{Re}\left(\frac{(5 - i2)e^{i3t}}{5^2 + 2^2}\right)\\
&= -\tfrac{2}{25+4}\,\mathrm{Re}\big((5 - i2)e^{i3t}\big) = -\tfrac{2}{29}\big(5\cos(3t) + 2\sin(3t)\big)\,.
\end{aligned}$$

Because the second forcing term is $-7e^{4t}$, we write

$$\mathrm{L}\left(\frac{-7e^{4t}}{-21}\right) = -7e^{4t}\,,$$

which implies that

$$x_{P2}(t) = \tfrac{1}{3}e^{4t}\,.$$

Therefore a general solution of the equation is

$$\begin{aligned}
x(t) &= x_H(t) + x_{P1}(t) + x_{P2}(t)\\
&= c_1 e^{-3t} + c_2 e^{7t} - \tfrac{10}{29}\cos(3t) - \tfrac{4}{29}\sin(3t) + \tfrac{1}{3}e^{4t}\,.
\end{aligned}$$

Undetermined Coefficients. Because $\mu + i\nu = i3$ and $m + d = m = 0$ for the forcing term $12\cos(3t)$, we *seek* a particular solution in the form

$$x_{P1}(t) = A\cos(3t) + B\sin(3t) .$$

Because

$$x'_{P1}(t) = -3A\sin(3t) + 3B\cos(3t) , \qquad x''_{P1}(t) = -9A\cos(3t) - 9B\sin(3t) ,$$

we see that

$$\begin{aligned} \mathrm{L}x_{P1}(t) &= x''_{P1}(t) - 4x'_{P1}(t) - 21x_{P1}(t) \\ &= \big[-9A\cos(3t) - 9B\sin(3t)\big] - 4\big[-3A\sin(3t) + 3B\cos(3t)\big] \\ &\quad - 21\big[A\cos(3t) + B\sin(3t)\big] \\ &= (-30A - 12B)\cos(3t) + (12A - 30B)\sin(3t) . \end{aligned}$$

By setting $\mathrm{L}x_{P1}(t) = 12\cos(3t)$, we see that

$$-30A - 12B = 12 , \qquad 12A - 30B = 0 .$$

The second equation implies $A = \frac{5}{2}B$, which when placed into the first equation yields $-87B = 12$. Hence, $B = -\frac{4}{29}$ and $A = -\frac{10}{29}$, which gives

$$x_{P1}(t) = -\tfrac{10}{29}\cos(3t) - \tfrac{4}{29}\sin(3t) .$$

Because $\mu + i\nu = 4$ and $m + d = m = 0$ for the forcing term $-7e^{4t}$, we *seek* a particular solution in the form

$$x_{P2}(t) = Ae^{4t} .$$

Because

$$x'_{P2}(t) = 4Ae^{4t} , \qquad x''_{P2}(t) = 16Ae^{4t} ,$$

we see that

$$\begin{aligned} \mathrm{L}x_{P2}(t) &= x''_{P2}(t) - 4x'_{P2}(t) - 21x_{P2}(t) \\ &= \big[16Ae^{4t}\big] - 4\big[4Ae^{4t}\big] - 21\big[Ae^{4t}\big] = (16 - 16 - 21)Ae^{4t} = -21Ae^{4t} . \end{aligned}$$

By setting $\mathrm{L}x_{P2}(t) = -7e^{4t}$, we see that $-21A = -7$, whereby $A = \frac{1}{3}$. Hence,

$$x_{P2}(t) = \tfrac{1}{3}e^{4t} .$$

Therefore a general solution of the equation is

$$\begin{aligned} x(t) &= x_H(t) + x_{P1}(t) + x_{P2}(t) \\ &= c_1e^{-3t} + c_2e^{7t} - \tfrac{10}{29}\cos(3t) - \tfrac{4}{29}\sin(3t) + \tfrac{1}{3}e^{4t} . \end{aligned}$$

(5) [10] What answer will be produced by the following MATLAB commands?

```
>> ode = 'D2y – 4*Dy + 13*y = 9*exp(5*t)';
>> dsolve(ode, 't')

ans =
```

Solution. The commands ask MATLAB to give a general solution of the equation

$$\mathrm{D}^2 y - 4\mathrm{D}y + 13y = 9e^{5t}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

This is a nonhomogeneous linear equation with constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 4z + 13 = (z-2)^2 + 9 = (z-2)^2 + 3^2\,,$$

which has the conjugate pair of roots $2 \pm i3$. Therefore a general solution of the associated homogeneous equation is

$$y_H(t) = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t)\,.$$

The forcing $9e^{5t}$ has degree $d = 0$ and characteristic $\mu + i\nu = 5$, which is a root of $p(z)$ of multiplicity $m = 0$. Therefore we can use either Key Identity Evaluations or Undetermined Coefficients to find a particular solution $y_P(t)$.

Key Indentity Evaluations. Because $m + d = m = 0$ we only need the Key identity, which is

$$\mathrm{L}\big(e^{zt}\big) = p(z)e^{zt} = (z^2 - 4z + 13)e^{zt}\,.$$

By evaluating this at $z = 5$ we obtain

$$\mathrm{L}\big(e^{5t}\big) = (5^2 - 4\cdot 5 + 13)e^{5t} = (25 - 20 + 13)e^{5t} = 18e^{5t}\,.$$

Because the forcing term is $9e^{5t}$, we see that a particular solution is $y_P(t) = \frac{1}{2}e^{5t}$. Therefore a general solution is

$$y(t) = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t) + \tfrac{1}{2}e^{5t}\,.$$

Undetermined Coefficients. Because $\mu + i\nu = 5$ and $m + d = m = 0$ for the forcing term $9e^{5t}$, we *seek* a particular solution in the form

$$y_P(t) = Ae^{5t}\,.$$

Because $y_P'(t) = 5Ae^{5t}$ and $y_P''(t) = 25Ae^{5t}$, we see that

$$\begin{aligned}\mathrm{L}y_P(t) &= y_P''(t) - 4y_P'(t) + 13y_P(t)\\ &= \big[25Ae^{5t}\big] - 4\big[5Ae^{5t}\big] + 13\big[Ae^{5t}\big]\\ &= (25 - 20 + 13)Ae^{5t} = 18Ae^{5t}\,.\end{aligned}$$

By setting $\mathrm{L}y_P(t) = 9e^{5t}$, we see that $18A = 9$, whereby $A = \frac{1}{2}$. Hence, we obtain the particular solution $y_P(t) = \frac{1}{2}e^{5t}$. Therefore a general solution is

$$y(t) = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t) + \tfrac{1}{2}e^{5t}\,.$$

(6) [8] Compute the Green function associated with the differential operator

$$\mathrm{L} = \mathrm{D}^2 - 8\mathrm{D} + 16\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. Because the differential operator L has constant coefficients, the Green function $g(t)$ associated with it satisfies the initial-value problem

$$\mathrm{L}g = \mathrm{D}^2 g - 8\mathrm{D}g + 16g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The characteristic polynomial is

$$p(z) = z^2 - 8z + 16 = (z-4)^2\,,$$

which has the double real root 4. Hence, the Green function has the form

$$g(t) = c_1 e^{4t} + c_2 t\, e^{4t}\,.$$

The initial condition $g(0) = 0$ implies that $c_1 = 0$. Because

$$g'(t) = 4c_2 t\, e^{4t} + c_2 e^{4t}\,,$$

the initial condition $g'(0) = 1$ implies that $c_2 = 1$. Therefore the Green function is

$$g(t) = t\, e^{4t}\,.$$

(7) [8] Solve the initial-value problem

$$v'' - 8v' + 16v = \frac{e^{4t}}{1+t}\,, \qquad v(0) = v'(0) = 0\,.$$

Solution. By the previous problem the Green function for this problem is $g(t) = t\, e^{4t}$. Because this equation is in normal form, the solution to this inital-value problem is given by the Green function formula

$$\begin{aligned} v(t) &= \int_0^t g(t-s) f(s)\,\mathrm{d}s = \int_0^t (t-s) e^{4(t-s)}\, \frac{e^{4s}}{1+s}\,\mathrm{d}s \\ &= t\, e^{4t} \int_0^t \frac{1}{1+s}\,\mathrm{d}s - e^{4t} \int_0^t \frac{s}{1+s}\,\mathrm{d}s\,. \end{aligned}$$

Because

$$\begin{aligned} &\int_0^t \frac{1}{1+s}\,\mathrm{d}s = \left[\log(1+s)\right]\Big|_{s=0}^t = \log(1+t)\,, \\ &\int_0^t \frac{s}{1+s}\,\mathrm{d}s = \int_0^t 1 - \frac{1}{1+s}\,\mathrm{d}s = \left[s - \log(1+s)\right]\Big|_{s=0}^t = t - \log(1+t)\,, \end{aligned}$$

we find that

$$\begin{aligned} v(t) &= t\, e^{4t} \log(1+t) - e^{4t}\big(t - \log(1+t)\big) \\ &= e^{4t}\big((1+t)\log(1+t) - t\big)\,. \end{aligned}$$

(8) [8] Find a particular solution $x_P(t)$ of the equation $x'' - x = 4e^t$.

Solution. This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 1 = (z+1)(z-1)\,,$$

which has two simple real roots -1 and 1. The forcing $4e^t$ has degree $d = 0$ and characteristic $\mu + i\nu = 1$, which is a root of $p(z)$ of multiplicity $m = 1$. Therefore we can use either Key Identity Evaluations or Undetermined Coefficients to find a particular solution $x_P(t)$.

Key Indentity Evaluations. Because $m = 1$ and $m + d = 1$ for the forcing term, we only need the first derivative of the Key Identity. The Key Identity and its first derivative are

$$\mathrm{L}\big(e^{zt}\big) = (z^2 - 1)e^{zt}\,,$$

$$\mathrm{L}\big(t\,e^{zt}\big) = (z^2 - 1)t\,e^{zt} + 2ze^{zt}\,.$$

By evaluating the derivative of the Key identity at $z = 1$ we obtain

$$\mathrm{L}\big(t\,e^{t}\big) = 2 \cdot 1 \cdot e^{1\cdot t} = 2e^t\,.$$

Because the forcing is $4e^t$, we see that a particular solution is $x_P(t) = 2t\,e^t$.

Undetermined Coefficients. Because $\mu + i\nu = 1$ and $m + d = m = 1$ for the forcing term $2e^t$, we *seek* a particular solution in the form

$$x_P(t) = At\,e^t\,.$$

Because

$$x_P'(t) = At\,e^t + Ae^t\,, \qquad x_P''(t) = At\,e^t + 2Ae^t\,,$$

we obtain

$$\mathrm{L}x_P(t) = \big[At\,e^t + 2Ae^t\big] - \big[At\,e^t\big] = 2Ae^t\,.$$

By setting $\mathrm{L}x_P(t) = 4e^t$, we see that $2A = 4$, whereby $A = 2$. Therefore, a particular solution is $x_P(t) = 2t\,e^t$.

Remark. A general solution is

$$x(t) = c_1e^t + c_2e^{-t} + 2t\,e^t\,.$$

(9) [10] Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2+i4$, $-2+i4$, $-2-i4$, $-2-i4$, $i3$, $-i3$, -7, -7, -7, 0, 0.
(a) Give the order of L.
(b) Give a general real solution of the homogeneous equation $\mathrm{L}y = 0$.

Solution (a). There are 11 roots listed above, so the degree of the characteristic polynomial is 11, whereby the order of L is also 11.

Solution (b). A general solution is

$$\begin{aligned} y(t) &= c_1 e^{2t}\cos(4t) + c_2 e^{2t}\sin(4t) + c_3 t\, e^{2t}\cos(4t) + c_4 t\, e^{2t}\sin(4t) \\ &\quad + c_5\cos(3t) + c_6\sin(3t) + c_7 e^{-7t} + c_8 t e^{-7t} + c_9 t^2 e^{-7t} + c_{10} + c_{11} t\,. \end{aligned}$$

Here the fundamental set of solutions is generated as follows:

- the double conjugate pair $2 \pm i4$ yields

$$e^{2t}\cos(4t)\,, \quad e^{2t}\sin(4t)\,, \quad t\, e^{2t}\cos(4t)\,, \quad \text{and} \quad t\, e^{2t}\sin(4t)\,;$$

- the single conjugate pair $\pm i3$ yields $\cos(3t)$ and $\sin(3t)$;
- the triple real root -7 yields e^{-7t}, $t\, e^{-7t}$, and $t^2 e^{-7t}$;
- the double real root 0 yields 1 and t.

(10) [10] The functions t^2 and t^3 are solutions of the homogeneous equation

$$t^2 y'' - 4ty' + 6y = 0 \qquad \text{over } t > 0\,.$$

(You do not have to check that this is true!)
(a) Show they are linearly independent.
(b) Give a general solution of the nonhomogeneous equation

$$t^2 y'' - 4ty' + 6y = \frac{t^3}{1+t^2} \qquad \text{over } t > 0\,.$$

Solution (a). The Wronskian of t^2 and t^3 is

$$W\big[t^2, t^3\big](t) = \det\begin{pmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{pmatrix} = t^2 \cdot 3t^2 - 2t \cdot t^3 = 3t^4 - 2t^4 = t^4\,.$$

Since $W[t^2, t^3](t) = t^4 \neq 0$ for $t > 0$, the functions t^2 and t^3 are linearly independent.

Solution (b). Because t^2 and t^3 are linearly independent, a general solution of the associated homogeneous problem is

$$y_H(t) = c_1 t^2 + c_2 t^3\,.$$

Because this problem has variable coefficients, you should use either the general Green Function method or Variation of Parameters to find a particular solution $y_P(t)$. Both of these methods require the equation to be put into its normal form, which is

$$y'' - \frac{4}{t}\, y' + \frac{6}{t^2}\, y = \frac{t}{1+t^2}\,.$$

General Green Function. The Green function $G(t,s)$ is given by

$$G(t,s) = \frac{1}{W[s^2, s^3](s)} \det\begin{pmatrix} s^2 & s^3 \\ t^2 & t^3 \end{pmatrix} = \frac{s^2 t^3 - t^2 s^3}{s^4} = \frac{t^3}{s^2} - \frac{t^2}{s}\,.$$

The Green function formula with $t_I = 1$ then yields the solution

$$y_P(t) = \int_1^t G(t,s)\, f(s)\, \mathrm{d}s = \int_1^t \left(\frac{t^3}{s^2} - \frac{t^2}{s} \right) \frac{s}{1+s^2}\, \mathrm{d}s$$
$$= t^3 \int_1^t \frac{1}{s(1+s^2)}\, \mathrm{d}s - t^2 \int_1^t \frac{1}{1+s^2}\, \mathrm{d}s\,.$$

We can evaluate the above definite integrals as

$$\int_1^t \frac{1}{s(1+s^2)}\, \mathrm{d}s = \int_1^t \frac{1}{s} - \frac{s}{1+s^2}\, \mathrm{d}s = \left(\log(s) - \tfrac{1}{2}\log(1+s^2)\right) \Big|_{s=1}^{t}$$
$$= \log(t) - \tfrac{1}{2}\log(1+t^2) + \tfrac{1}{2}\log(2) = \tfrac{1}{2}\log\!\left(\frac{2t^2}{1+t^2} \right),$$
$$\int_1^t \frac{1}{1+s^2}\, \mathrm{d}s = \tan^{-1}(s)\Big|_{s=1}^{t} = \tan^{-1}(t) - \tan^{-1}(1) = \tan^{-1}(t) - \tfrac{\pi}{4}\,.$$

Therefore a general solution is

$$y(t) = c_1 t^2 + c_2 t^3 + \tfrac{1}{2} t^3 \log\!\left(\frac{2t^2}{1+t^2} \right) - t^2 \left(\tan^{-1}(t) - \tfrac{\pi}{4} \right).$$

Variation of Parameters. We *seek* a solution in the form

$$y(t) = u_1(t) t^2 + u_2(t) t^3\,.$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$u_1'(t) t^2 + u_2'(t) t^3 = 0\,, \qquad u_1'(t) 2t + u_2'(t) 3t^2 = \frac{t}{1+t^2}\,.$$

The solution of this system is

$$u_1'(t) = -\frac{1}{1+t^2}\,, \qquad u_2'(t) = \frac{1}{t(1+t^2)}\,.$$

Alternatively, because $W[t^2, t^3](t) = t^4$, the formulas from the book yield

$$u_1'(t) = -\frac{t^3}{t^4}\,\frac{t}{1+t^2} = -\frac{1}{1+t^2}\,, \qquad u_2'(t) = \frac{t^2}{t^4}\,\frac{t}{1+t^2} = \frac{1}{t(1+t^2)}\,.$$

No matter how they are obtained, you integrate these equations to find

$$u_1(t) = -\int \frac{1}{1+t^2}\, \mathrm{d}t = c_1 - \tan^{-1}(t)\,,$$
$$u_2(t) = \int \frac{1}{t(1+t^2)}\, \mathrm{d}t = \int \frac{1}{t} - \frac{t}{1+t^2}\, \mathrm{d}t$$
$$= c_2 + \log(t) - \tfrac{1}{2}\log(1+t^2) = c_2 + \tfrac{1}{2}\log\!\left(\frac{t^2}{1+t^2} \right).$$

Therefore a general solution is

$$y(t) = \left(c_1 - \tan^{-1}(t)\right) t^2 + \left(c_2 + \tfrac{1}{2}\log\!\left(\frac{t^2}{1+t^2} \right) \right) t^3$$
$$= c_1 t^2 + c_2 t^3 + \tfrac{1}{2} t^3 \log\!\left(\frac{t^2}{1+t^2} \right) - t^2 \tan^{-1}(t)\,.$$

(11) [10] The vertical displacement of an unforced mass on a spring is given by

$$h(t) = -4\cos(\pi t) - 3\sin(\pi t)\,.$$

(a) Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)
(b) Express $h(t)$ in the amplitude-phase form $h(t) = A\cos(\pi t - \delta)$ with $A > 0$ and $0 \leq \delta < 2\pi$. Label the amplitude and phase. (The phase may be expressed in terms of an inverse trig function.)
(c) Give the natural frequency and period of this spring-mass system.
(d) If an external force $F_{ext}(t) = 5\cos(\omega t)$ is applied, at what value of the driving frequency ω does resonance occur?

Solution (a). The system is undamped because the given displacement $h(t)$ comes from an underlying characteristic polynomial having the conjugate pair of roots $\pm i\pi$.

Solution (b). By comparing

$$A\cos(\pi t - \delta) = A\cos(\delta)\cos(\pi t) + A\sin(\delta)\sin(\pi t)\,,$$

with $h(t) = -4\cos(\pi t) - 3\sin(\pi t)$, we see that

$$A\cos(\delta) = -4\,, \qquad A\sin(\delta) = -3\,.$$

This shows that (A, δ) are the polar coordinates of the point in the plane whose Cartesian coordinates are $(-4, -3)$. Clearly A is given by

$$A = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5\,.$$

Because $(-4, -3)$ lies in the *third quadrant*, the phase δ satisfies $\pi < \delta < \frac{3}{2}\pi$. There are several ways to express δ. A picture shows that if we use π as a reference then

$$\sin(\delta - \pi) = \tfrac{3}{5}\,, \qquad \tan(\delta - \pi) = \tfrac{3}{4}\,, \qquad \cos(\delta - \pi) = \tfrac{4}{5}\,,$$

and we can express the phase by any one of the formulas

$$\delta = \pi + \sin^{-1}\big(\tfrac{3}{5}\big)\,, \qquad \delta = \pi + \tan^{-1}\big(\tfrac{3}{4}\big)\,, \qquad \delta = \pi + \cos^{-1}\big(\tfrac{4}{5}\big)\,,$$

The same picture shows that if we use $\frac{3}{2}\pi$ as a reference then

$$\sin(\tfrac{3}{2}\pi - \delta) = \tfrac{4}{5}\,, \qquad \tan(\tfrac{3}{2}\pi - \delta) = \tfrac{4}{3}\,, \qquad \cos(\tfrac{3}{2}\pi - \delta) = \tfrac{3}{5}\,,$$

and we can express the phase by any one of the formulas

$$\delta = \tfrac{3}{2}\pi - \sin^{-1}\big(\tfrac{4}{5}\big)\,, \qquad \delta = \tfrac{3}{2}\pi - \tan^{-1}\big(\tfrac{4}{3}\big)\,, \qquad \delta = \tfrac{3}{2}\pi - \cos^{-1}\big(\tfrac{3}{5}\big)\,.$$

Only one expression for δ is required.

Remark. It is incorrect to give the phase by one of the formulas

$$\delta = \sin^{-1}\big(-\tfrac{3}{5}\big)\,, \qquad \delta = \tan^{-1}\big(\tfrac{3}{4}\big)\,, \qquad \delta = \cos^{-1}\big(-\tfrac{4}{5}\big)\,,$$

because, by our conventions for the range of the inverse trigonometric functions, $\sin^{-1}\big(-\frac{3}{5}\big)$ lies in $(-\frac{\pi}{2}, 0)$, $\tan^{-1}\big(\frac{3}{4}\big)$ lies in $(0, \frac{\pi}{2})$, while $\cos^{-1}\big(-\frac{4}{5}\big)$ lies in $(\frac{\pi}{2}, \pi)$.

Solution (c). The natural frequency ω_o is π, so the natural period T_o is given by

$$T_o = \frac{2\pi}{\omega_o} = \frac{2\pi}{\pi} = 2 \quad \text{sec}\,.$$

Solution (d). Resonance occurs when the driving frequency ω equals the natural fequency ω_o. Given the answer to part (c), resonance occurs when

$$\omega = \omega_o = \pi \quad 1/\text{sec}\,.$$

(12) [8] When a 5 gram mass is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) At $t = 0$ the mass is displaced 7 cm below its rest position and is released with a downward velocity of 3 cm/sec. The medium imparts a damping force of 120 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 2 cm/sec. Assume that the spring force is proportional to displacement, that the damping is proportional to veloicity, and that there are no other forces.

(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

(b) What is the quasifrequency ν of this spring?

Solution (a). Let $h(t)$ be the displacement (in centimeters) of the mass from its rest position at time t (in seconds), with upward displacements being positive. The governing initial-value problem then has the form

$$mh'' + \gamma h' + kh = 0\,, \qquad h(0) = -7\,, \quad h'(0) = -3\,,$$

where m is the mass, γ is the damping coefficient, and k is the spring constant. We are given that $m = 3$ grams. We obtain k by balancing the force applied by the spring when it is stetched 9.8 cm with the weight of the mass ($mg = 5 \cdot 980$ dynes). This gives $k\,9.8 = 5 \cdot 980$, or

$$k = \frac{5 \cdot 980}{9.8} = 500 \text{ dynes/cm}\,.$$

We obtain γ by balancing the damping force when the speed of the mass is 2 cm/sec with 120 dynes. This gives $\gamma\,2 = 120$, or

$$\gamma = \frac{120}{2} = 60 \quad \text{dynes sec/cm}$$

Therefore the governing initial-value problem is

$$5h'' + 60h' + 500h\,, \qquad h(0) = -7\,, \quad h'(0) = -3\,.$$

Solution (b). The normal form of the governing equation is

$$h'' + 12h' + 100h\,.$$

Its charactericitc polynomial is

$$p(z) = z^2 + 12z + 100 = (z+6)^2 + 64 = (z+6)^2 + 8^2\,.$$

This has conjugate roots $-6 \pm i8$. Therefore the quasifrequency is given by $\nu = 8$ 1/sec.