First In-Class Exam Solutions
Math 246, Professor David Levermore
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(1) [12] Consider the differential equation $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{(z^2-9)(9-z)}{9+z^2}$.

(a) [6] Sketch its phase-line portrait over the interval $-6 \le z \le 12$. Identify all of its stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.

(b) [2] If $z(0) = 6$, how does the solution $z(t)$ behave as $t \to \infty$?

(c) [2] If $z(0) = 0$, how does the solution $z(t)$ behave as $t \to \infty$?

(d) [2] If $z(0) = -5$, how does the solution $z(t)$ behave as $t \to \infty$?

Solution (a). The right-hand side of the equation has the positive denominator $9+z^2$ and a numerator that factors as $(z+3)(z-3)(9-z)$. Therefore the stationary solutions are $z=-3$, $z=3$, and $z=9$. A sign analysis shows that the phase-line for this equation is

$$\begin{array}{ccccccc} + & & - & & + & & - \\ \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow \; z \\ & -3 & & 3 & & 9 & \\ & \text{stable} & & \text{unstable} & & \text{stable} & \end{array}$$

Solution (b). The phase-line shows that if $z(0) = 6$ then $z(t) \to 9$ as $t \to \infty$.
Solution (c). The phase-line shows that if $z(0) = 0$ then $z(t) \to -3$ as $t \to \infty$.
Solution (d). The phase-line shows that if $z(0) = -5$ then $z(t) \to -3$ as $t \to \infty$.

(2) [22] Find the explicit solution for each of the following initial-value problems and identify its interval of definition.

(a) $\dfrac{\mathrm{d}w}{\mathrm{d}t} = \dfrac{\cos(t) - 2tw}{t^2}, \quad w(\pi) = 0.$

Solution. This equation is linear. Its normal form is

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{2}{t}\,w = \frac{\cos(t)}{t^2}\,.$$

An integrating factor is

$$\exp\left(\int_1^t \frac{2}{s}\,\mathrm{d}s\right) = \exp(2\log(t)) = t^2\,,$$

so that it has the integrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^2 w) = t^2 \cdot \frac{\cos(t)}{t^2} = \cos(t)\,, \qquad \Longrightarrow \qquad t^2 w = \sin(t) + c\,.$$

The initial condition $w(\pi) = 0$ implies that $c = \pi^2 \cdot 0 - \sin(\pi) = 0 - 0 = 0$. Therefore

$$w = \frac{\sin(t)}{t^2}\,, \quad \text{with interval of definition } 0 < t < \infty\,.$$

Remark. The interval of definition can be read off from the normal form of the equation because both the coefficient and forcing are continuous everywhere except at $t = 0$, and the initial time is $t = \pi$.

(b) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = t^2 e^{-3x}, \quad x(0) = x_I.$

Solution. This equation is separable. It has no stationary points. Its separated differential form is $e^{3x}\,\mathrm{d}x = t^3\,\mathrm{d}t$, whereby

$$\int e^{3x}\,\mathrm{d}x = \int t^2\,dt\,.$$

By integrating both sides gives $\frac{1}{3}e^{3x} = \frac{1}{3}t^3 + c$. The initial condition then implies that $\frac{1}{3}e^{3x_I} = \frac{1}{3}0^3 + c$, whereby $c = \frac{1}{3}e^{3x_I}$. The solution is thereby given implicity by

$$e^{3x} = t^3 + e^{3x_I}\,.$$

Whenever $t^3 + e^{3x_I} > 0$ this can be solved explicitly for x to obtain

$$x = \tfrac{1}{3}\log\big(t^3 + e^{3x_I}\big)\,, \quad \text{with interval of definition } -e^{x_I} < t < \infty\,.$$

Remark. The condition $t^3 + e^{3x_I} > 0$ implies that $t^3 > -e^{3x_I}$, which implies that $t > -e^{x_I}$. The interval of definition for this solution also can be given either with the interval notation $(-e^{x_I}, \infty)$ or by the inequality $-e^{x_I} < t$.

(3) [6] Give the interval of definition for the solution of the initial-value problem

$$\sin(t)\,\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{u}{t^2 - 16} = \frac{1}{t^2 - 4}\,, \qquad u(5) = 2\,.$$

(You do not have to solve this equation to answer this question!)

Solution. This problem is linear in u. Its normal form is

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{(t^2 - 16)\sin(t)}\,u = \frac{1}{(t^2 - 4)\sin(t)}\,.$$

The coefficient $1/((t^2 - 16)\sin(t))$ is continuous everywhere except at $t = \pm 4$ and $t = n\pi$ for every integer n, while the forcing $1/((t^2-4)\sin(t))$ is continuous everywhere except at $t = \pm 2$ and $t = n\pi$ for every integer n. Therefore you can see that the interval of definition is $(4, 2\pi)$, which is the largest interval containing the initial time $t = 5$ that does not contain the points $t = \pm 2$, $t = \pm 4$, or $t = n\pi$ for any integer n.

(4) [8] Suppose you have used a numerical method to approximate the solution of an initial-value problem over the time interval $[0, 5]$ with 1000 uniform time steps. About how many uniform time steps do you need to reduce the global error of your approximation by a factor of $\frac{1}{625}$ if the method you had used was each of the following?

(a) Explicit Euler method

Solution. The explicit Euler method is first order, so its error scales like h. To reduce the error by a factor of $\frac{1}{625}$, you must reduce h by a factor of $\frac{1}{625}$. You must increase the number of time steps by a factor of 625, which means you need $625,000$ uniform time steps.

(b) Runge-midpoint method

Solution. The Runge-midpoint method is second order, so its error scales like h^2. To reduce the error by a factor of $\frac{1}{625}$, you must reduce h by a factor of $\frac{1}{625}^{\frac{1}{2}} = \frac{1}{25}$. You must increase the number of time steps by a factor of 25, which means you need $25,000$ uniform time steps.

(c) Runge-trapezoidal method

Solution. The Runge-trapezoidal method is second order, so its error scales like h^2. To reduce the error by a factor of $\frac{1}{625}$, you must reduce h by a factor of $\frac{1}{625}^{\frac{1}{2}} = \frac{1}{25}$. You must increase the number of time steps by a factor of 25, which means you need $25,000$ uniform time steps.

(d) Runge-Kutta method

Solution. The Runge-Kutta method is fourth order, so its error scales like h^4. To reduce the error by a factor of $\frac{1}{625}$, you must reduce h by a factor of $\frac{1}{625}^{\frac{1}{4}} = \frac{1}{5}$. You must increase the number of time steps by a factor of 5, which means you need $5,000$ uniform time steps.

(5) [16] Consider the following MATLAB function M-file.

```
function [t,y] = solveit(tI, yI, tF, n)

t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = tI; y(1) = yI; h = (tF - tI)/n; hhalf = h/2;
for j = 1:n
fnow = (t(j))^2 − exp(t(j)*y(j));
thalf = t(j) + hhalf; yhalf = y(j) + hhalf*fnow;
fhalf = (thalf)^2 − exp(thalf*yhalf);
t(j + 1) = t(j) + h; y(j + 1) = y(j) + h*fhalf;
end
```

Suppose the input values are tI = 0, yI = 1, tF = 5, and n = 250.

(a) [4] What initial-value problem is being approximated numerically?
(b) [2] What is the numerical method being used?
(c) [2] What is the timestep?
(d) [8] What will be the output values of t(2) and y(2)?

Solution (a). The initial-value problem being approximated numerically is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 - e^{ty}\,, \qquad y(0) = 1\,.$$

Remark. An initial-value problem consists of both a differential equation and an initial condition. Without both you did not get full credit.

Solution (b). The Runge-midpoint (modified Euler) method is being used. (This is clear from the last line in the "for" loop.)

Solution (c). Because tF = 5, tI = 0, and n = 250, the time step is

$$\mathrm{h} = \frac{\mathrm{tF} - \mathrm{tI}}{\mathrm{n}} = \frac{5-0}{250} = \frac{1}{50} = .02\,.$$

Remark. You had to plug in the correct values for tF, tI, and n to get any credit.

Solution (d). Because h = .02, we have hhalf = .01.
Because tI = 0 and yI = 1, we have t(1) = tI = 0, and y(1) = yI = 1.
Setting j = 1 inside the "for" loop then yields

$$\begin{aligned}
&\mathrm{fnow} = (\mathrm{t}(1))^2 - \exp(\mathrm{t}(1)^*\mathrm{y}(1)) = 0^2 - e^{0\cdot 1} = 0 - 1 = -1\,,\\
&\mathrm{thalf} = \mathrm{t}(1) + \mathrm{hhalf} = 0 + .01 = .01\,,\\
&\mathrm{yhalf} = \mathrm{y}(1) + \mathrm{hhalf} * \mathrm{fnow} = 1 + .01\cdot(-1) = 1 - .01 = .99\,,\\
&\mathrm{fhalf} = \mathrm{thalf}^2 - \exp(\mathrm{thalf}^*\mathrm{yhalf}) = (.01)^2 - e^{.01\cdot .99} = .0001 - e^{.0099}\,,\\
&\mathrm{t}(2) = \mathrm{t}(1) + \mathrm{h} = 0 + .02 = .02\,,\\
&\mathrm{y}(2) = \mathrm{y}(1) + \mathrm{h} * \mathrm{fhalf} = 1 + .02\,(.0001 - e^{.0099})\,.
\end{aligned}$$

Remark. You did not have to simplify this expression for y(2) to get full credit. At best, you could have simplified it to y(2) = 1.000002 − .02 $e^{.0099}$.

(6) [12] In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population such that it would triple every five weeks. There are 180,000 mosquitoes in the area when a flock of birds arrives that eats 45,000 mosquitoes per week.

(a) [6] Write down an initial-value problem that governs $M(t)$, the population of mosquitoes in the area after the flock of birds arrives.

(b) [6] How many mosquitoes are left after the flock has been there two weeks?

Solution (a). The population tripling every five weeks corresponds to a growth factor of $3^{\frac{t}{5}} = (e^{\log(3)})^{\frac{t}{5}} = e^{\frac{1}{5}\log(3)t}$, which implies a growth rate of $\frac{1}{5}\log(3)$. Therefore the initial-value problem that M satisfies is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \tfrac{1}{5}\log(3)M - 45{,}000\,, \qquad M(0) = 180{,}000\,.$$

Solution (b). This equation is linear. Its normal form is

$$\frac{\mathrm{d}M}{\mathrm{d}t} - \tfrac{1}{5}\log(3)M = -45{,}000\,.$$

An integrating factor is $e^{-\frac{1}{5}\log(3)t}$, so that it has the integrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{1}{5}\log(3)t}M\right) = -45{,}000e^{-\frac{1}{5}\log(3)t}\,.$$

Integrating this equation gives

$$e^{-\frac{1}{5}\log(3)t}M(t) = \frac{5\cdot 45{,}000}{\log(3)}e^{-\frac{1}{5}\log(3)t} + c\,.$$

Then the initial condition $M(0) = 180{,}000$ implies

$$c = e^{-\frac{1}{5}\log(3)0}180{,}000 - \frac{5\cdot 45{,}000}{\log(3)}e^{-\frac{1}{5}\log(3)0} = 180{,}000 - \frac{5\cdot 45{,}000}{\log(3)}\,.$$

Therefore the number of mosquitoes left after the flock has been there two weeks is

$$M(2) = \frac{5 \cdot 45,000}{\log(3)} + \left(180,000 - \frac{5 \cdot 45,000}{\log(3)}\right) e^{\frac{2}{5}\log(3)} .$$

You did not have to simplify. At best you can simplify it to

$$M(2) = \frac{225,000}{\log(3)} + \left(180,000 - \frac{225,000}{\log(3)}\right) 9^{\frac{1}{5}} .$$

(7) [24] Give an implicit general solution to each of the following differential equations.

(a) $(ye^{xy} - \sin(x))\,\mathrm{d}x + (xe^{xy} + \cos(y))\,\mathrm{d}y = 0$.

Solution. This differential form is *exact* because

$$\partial_y(ye^{xy} - \sin(x)) = e^{xy} + xye^{xy} \quad = \quad \partial_x(xe^{xy} + \cos(y)) = e^{xy} + xye^{xy} .$$

Therefore we can find $H(x,y)$ such that

$$\partial_x H(x,y) = ye^{xy} - \sin(x) , \qquad \partial_y H(x,y) = xe^{xy} + \cos(y) .$$

Integrating the first equation with respect to x yields

$$H(x,y) = e^{xy} + \cos(x) + h(y) ,$$

whereby

$$\partial_y H(x,y) = xe^{xy} + h'(y) .$$

Plugging this expression for $\partial_y H(x,y)$ into the second equation gives

$$xe^{xy} + h'(y) = xe^{xy} + \cos(y) ,$$

which yields $h'(y) = \cos(y)$. Taking $h(y) = \sin(y)$, an implicit general solution is given by

$$e^{xy} + \cos(x) + \sin(y) = c .$$

(b) $(y^2 + 3x^2y)\,\mathrm{d}x + (2x^3 + 3xy)\,\mathrm{d}y = 0$.

Solution. This differential form is *not exact* because

$$\partial_y(y^2 + 3x^2y) = 2y + 3x^2 \quad \neq \quad \partial_x(2x^3 + 3xy) = 6x^2 + 3y .$$

Therefore we seek an *integrating factor* μ such that

$$\partial_y[(y^2 + 3x^2y)\mu] = \partial_x[(2x^3 + 3xy)\mu] .$$

Expanding the partial derivatives yields

$$(y^2 + 3x^2y)\partial_y\mu + (2y + 3x^2)\mu = (2x^3 + 3xy)\partial_x\mu + (6x^2 + 3y)\mu .$$

Grouping the μ terms together gives

$$(y^2 + 3x^2y)\partial_y\mu = (2x^3 + 3xy)\partial_x\mu + (3x^2 + y)\mu .$$

If you set $\partial_x\mu = 0$ then this becomes

$$(y^2 + 3x^2y)\partial_y\mu = (3x^2 + y)\mu ,$$

which becomes

$$(y + 3x^2)y\partial_y\mu = (y + 3x^2)\mu ,$$

which reduces to $y\partial_y\mu = \mu$. This yields the integrating factor $\mu = y$.

Because y is an integrating factor, the differential form

$$(y^3 + 3x^2y^2)\,\mathrm{d}x + (2x^3y + 3xy^2)\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

Therefore we can find $H(x, y)$ such that

$$\partial_x H(x, y) = y^3 + 3x^2y^2\,, \qquad \partial_y H(x, y) = 2x^3y + 3xy^2\,.$$

Integrating the first equation with respect to x yields

$$H(x, y) = xy^3 + x^3y^2 + h(y)\,,$$

whereby

$$\partial_y H(x, y) = 3xy^2 + 2x^3y + h'(y)\,.$$

Plugging this expression for $\partial_y H(x, y)$ into the second equation gives

$$3xy^2 + 2x^3y + h'(y) = 2x^3y + 3xy^2\,,$$

which yields $h'(y) = 0$. Taking $h(y) = 0$, an implicit general solution is given by

$$xy^3 + x^3y^2 = c\,.$$