3. Properties of the Boltzmann Equation

The form of the collision operator (**??**) leads to a theory that is rich with structure, much of which does not depend on the details of the collision kernel b. Here we will indicate how some of that structure arises. In particular, we survey the basic properties relating to dilation, and Galilean symmetries; mass, momentum, and energy conservation; and entropy dissipation.

3.1. **Symmetries and Actions.**

The collision operator $\mathcal{B}$ generally has symmetries relating to dilations of f and to translations, reflections, and rotations of v. When the kernel b has the factorized form (**??**) then $\mathcal{B}$ has an additional symmetry relating to dilations of v. Each of these symmetries leads to an action on functions $F = F(v, x, t)$ defined over $(v, x, t) \in \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R}$ that leaves the set of solutions to the Boltzmann equation (**??**) invariant. Of course, the set of solutions to the Boltzmann equation is also invariant under space and time translations. These symmetries largely determine the form taken by fluid dynamical systems.

3.1.1. *Dilations of f.*

The operator $\mathcal{B}$ is quadratically homogeneous. This means that if $\mathcal{B}(f, f)$ makes sense for some $f = f(v)$ then it does so for the dilated function αf for every $\alpha \in \mathbb{R}_+$ with

$$\mathcal{B}(\alpha f, \alpha f) = \alpha^2 \mathcal{B}(f, f)\,. \tag{3.1}$$

This symmetry reflects the fact that $\mathcal{B}$ models binary collisions.

Now for every function $F = F(v, x, t)$ defined over $(v, x, t) \in \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R}$ and every $\alpha \in \mathbb{R}_+$ define the action $\mathcal{A}^Q_\alpha$ by

$$\mathcal{A}^Q_\alpha F = \mathcal{A}^Q_\alpha F(v.x, t) = \alpha\, F(v, \alpha x, \alpha t)\,. \tag{3.2}$$

The quadratic homogeneity of $\mathcal{B}$ (3.1) implies that if F satisfies the Boltzmann equation (**??**) then so does $\mathcal{A}^Q_\alpha F$ for every $\alpha \in \mathbb{R}_+$.

3.1.2. *Translations, Reflections, and Rotations of v.*

The operator $\mathcal{B}$ commutes with translational and orthogonal transformations. Specifically, given any $f = f(v)$ then for every vector $u \in \mathbb{R}^D$ and for every orthogonal matrix $O \in \mathbb{R}^{D\times D}$ define transformed functions $\mathcal{T}_u f$ and $\mathcal{T}_o f$ by

$$\mathcal{T}_u f(v) \equiv f(v - u)\,, \qquad \mathcal{T}_o f(v) \equiv f(O^T v)\,. \tag{3.3}$$

It follows from the definition of $\mathcal{B}$ and the form of b (**??**) that if $\mathcal{B}(f, f)$ makes sense for $f = f(v)$ then it does so for $\mathcal{T}_u f$ and $\mathcal{T}_o f$ with

$$\mathcal{B}(\mathcal{T}_u f, \mathcal{T}_u f) = \mathcal{T}_u \mathcal{B}(f, f)\,, \qquad \mathcal{B}(\mathcal{T}_o f, \mathcal{T}_o f) = \mathcal{T}_o \mathcal{B}(f, f)\,. \tag{3.4}$$

These symmetries reflect the Galilean invariance of the microscopic dynamics that underlies b.

For every function $F = F(v, x, t)$ defined over $(v, x, t) \in \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R}$, every $u \in \mathbb{R}^D$, and every orthogonal matrix $O \in \mathbb{R}^{D\times D}$ define the actions $\mathcal{A}^T_u$ and $\mathcal{A}^R_o$ by

$$\begin{aligned} \mathcal{A}^T_u F &= \mathcal{A}^T_u F(v, x, t) \equiv F(v - u, x - ut, t)\,, \\ \mathcal{A}^R_o F &= \mathcal{A}^R_o F(v, x, t) \equiv F(O^T v, O^T x, t)\,. \end{aligned} \tag{3.5}$$

The translational and orthogonal symetries (3.4) imply that if F satisfies the Boltzmann equation (**??**) then so does $\mathcal{A}^T_u F$ for every vector $u \in \mathbb{R}^D$, and so does $\mathcal{A}^R_o F$ for every orthogonal matrix $O \in \mathbb{R}^{D\times D}$ $u \in \mathbb{R}^D$.

3.1.3. *Dilations involving v.* The operator $\mathcal{B}$ commutes with a dilational transformation when the collision kernel b has the factorized form (**??**) for some $\beta \in (-D, 1]$. Specifically, given any $f = f(v)$ then for every $\lambda \in \mathbb{R}_+$ define tha transformed function $\mathcal{T}_\lambda f$ by

$$\mathcal{T}_\lambda f(v) \equiv \lambda^{\beta+D} f(\lambda v)\,. \tag{3.6}$$

It follows from the definition of $\mathcal{B}$ and the factorized form of b (**??**) that if $\mathcal{B}(f,f)$ makes sense for $f = f(v)$ then it does so for $\mathcal{T}_\lambda f$ with

$$\mathcal{B}(\mathcal{T}_\lambda f, \mathcal{T}_\lambda f) = \mathcal{T}_\lambda \mathcal{B}(f,f)\,. \tag{3.7}$$

This symmetry reflects a scale invariance of the microscopic dynamics that underlies b that occurs when the intermolecular potential has the form c/r^k.

For every function $F = F(v,x,t)$ defined over $(v,x,t) \in \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R}$ and every $\lambda \in \mathbb{R}_+$ define the action $\mathcal{A}^S_\lambda$ by

$$\mathcal{A}^S_\lambda F = \mathcal{A}^S_\lambda F(v.x,t) = \lambda^{\beta+D} F(\lambda v, \lambda x, t)\,. \tag{3.8}$$

The dilational symmetry (3.9) implies that if F satisfies the Boltzmann equation (**??**) then so does $\mathcal{A}^S_\lambda F$ for every $\lambda \in \mathbb{R}_+$.

3.2. **Boltzmann Identity.** In what follows the integral of any scalar-, vector-, or matrix-valued measurable function $f = f(v)$ over the D-dimensional Lebesgue measure $\mathrm{d}v$ will be denoted by $\langle f \rangle$; thus

$$\langle f \rangle = \int f(v)\, \mathrm{d}v\,. \tag{3.9}$$

All functions are understood to be Lebesgue measurable in all variables.

Much of the formal structure of the Boltzmann equation follows from an identity due to Boltzmann regarding the collision operator (**??**). It derives from the fact that the measure $b(\omega, v_* - v)\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v$ on $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$ is invariant under the transformations

$$\begin{aligned} (\omega, v_*, v) &\mapsto (\omega, v, v_*)\,, \\ (\omega, v_*, v) &\mapsto (\omega, v'_*, v')\,, \\ (\omega, v_*, v) &\mapsto (\omega, v', v'_*)\,. \end{aligned} \tag{3.10}$$

This invariance follows from the Galilean symmetry (**??**) and the fact the linear relationship (**??**) between (v_*, v) and (v'_*, v') satisfies (**??**). Repeated application of the invariant transformations (3.10) yields the Boltzmann identity:

$$\langle \xi\, \mathcal{B}(f,f) \rangle = \frac{1}{4} \iiint (\xi + \xi_* - \xi' - \xi'_*)\, (f'_* f' - f_* f)\, b\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v\,, \tag{3.11}$$

for every $\xi = \xi(v)$ and $f = f(v)$ for which the integrals make sense. There are many ways to show this. We outline one below.

Let f be any bounded measurable function of v with compact support and ξ be any locally bounded measurable function of v. These assumptions along with the assumption that b is locally integrable with respect to $\mathrm{d}\omega\, \mathrm{d}v_*$ insure the existence of the triple (Lebesgue) integrals

$$\iiint \xi\, f'_* f'\, b\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v\,, \qquad \iiint \xi\, f_* f\, b\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v\,,$$

and that one can apply the variable transformations (3.10) to these intgrals. By first using (**??**) to express $\langle \xi\, \mathcal{B}(f,f)\rangle$ as a triple integral and then applying each of the invariant transformations (3.10) to the resulting expression, one finds that

$$\begin{aligned}\langle \xi\, \mathcal{B}(f,f)\rangle &= \iiint \xi\,(f_*'f' - f_*f)\, b\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v \\ &= \iiint \xi_*\,(f_*'f' - f_*f)\, b\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v \\ &= -\iiint \xi'\,(f_*'f' - f_*f)\, b\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v \\ &= -\iiint \xi_*'\,(f_*'f' - f_*f)\, b\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v\,.\end{aligned}$$

One then obtains (3.11) by simply averaging the four expressions on the right-hand side above. Notice that here ξ can be either a scalar-, vector-, or matrix-valued function.

3.3. **Collision Invariants and Locally Conserved Quantities.** The second fundamental property of the Boltzmann collision operator is that the measure $b(\omega, v_* - v)\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v$ on $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$ characterizes the so-called *collision invariants* and *locally conserved quantities*.

Definition. *Collision invariants* are those measurable functions $\xi = \xi(v)$ that satisfy

$$\xi + \xi_* = \xi' + \xi_*' \quad \text{for almost every } (\omega, v_*, v)\,. \tag{3.12a}$$

Locally conserved quantities are those locally integrable functions $\xi = \xi(v)$ that satisfy

$$\langle \xi\, \mathcal{B}(f,f)\rangle = 0 \quad \text{for any integrable } f \text{ with compact support}\,. \tag{3.12b}$$

It is clear from (**??**,**??**) that every linear combination of $1, v_1, v_2, \cdots, v_D$, and $|v|^2$ is a collision invariant. It also follows directly from the Boltzmann identity (3.11) that every collision invariant is a locally conserved quantity. It is less clear that these implications also go the other way. Specifically, we have the following characterization.

Theorem. The following statements are equivalent:

$$\begin{aligned} &\text{(i)} \quad \xi \in \operatorname{span}\{1, v_1, v_2, \cdots, v_D, |v|^2\}\,; \\ &\text{(ii)} \quad \xi \text{ is a collision invariant}\,; \\ &\text{(iii)} \quad \xi \text{ is a locally conserved quantity}\,; \\ &\text{(iv)} \quad (\xi + \xi_*)\, \overline{b} = \int (\xi' + \xi_*')\, b\, \mathrm{d}\omega\,, \end{aligned} \tag{3.13}$$

where $\overline{b}(v_* - v)$ is defined by

$$\overline{b}(v_* - v) = \int b(\omega, v_* - v)\, \mathrm{d}\omega\,. \tag{3.14}$$

Remark. This characterization follows from the fact that b is positive almost everywhere with respect to the measure $\mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v$, but requires a bit of an argument.

Proof. It is fairly easy to see that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). Indeed, (i) $\Longrightarrow$ (ii) follows directly from the microscopic conservation laws (**??**,**??**) while (ii) $\Longrightarrow$ (iii) follows directly from the Boltzmann identity (3.11).

The fact that (iii) $\implies$ (iv) follows by first polarizing the Boltzmann identity to find that for every $f = f(v)$ and $g = g(v)$ of compact support one has

$$\langle \xi\, \mathcal{B}(f,g) \rangle = \frac{1}{8} \iiint (\xi + \xi_* - \xi' - \xi'_*) \\ (f' g'_* + f'_* g' - f g_* - f_* g)\, b \, \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v \,,$$

One then applies each of the invariant transformations (3.10) to obtain

$$\begin{aligned} 0 &= \langle \xi\, \mathcal{B}(f,g) \rangle \\ &= -\frac{1}{2} \iiint (\xi + \xi_* - \xi' - \xi'_*)\, f_* g \, b \, \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v \\ &= -\frac{1}{2} \iint \left((\xi + \xi_*)\, \overline{b} - \int (\xi' + \xi'_*)\, b \, \mathrm{d}\omega \right) f_* g \, \mathrm{d}v_* \, \mathrm{d}v \,. \end{aligned}$$

Then (iv) follows by envoking the arbitrariness of f and g.

To show that (iv) $\implies$ (i) requires more work. If you assume that ξ is either twice differentiable or just continuous, you can argue in the style of classical proofs of the fact that such a function that is additive is also linear. Roughly speaking, if ξ is smooth one argues as follows. First, observe that when ξ is a collision invariant it satisfies

$$\xi + \xi_* = \frac{1}{|\mathbb{S}^{D-1}|} \int \left(\xi' + \xi'_* \right) \mathrm{d}n' \,.$$

Next, argue that this equation has the functional form

$$\xi(v) + \xi(v_*) = G(v + v_*, |v|^2 + |v_*|^2) \,.$$

Finally, argue that the only solution of this equation is $\xi \in \mathrm{span}\{1, v_1, v_2, \cdots, v_D, |v|^2\}$.

The details of these two steps above are left as an exercise with the caution that they are not completely trivial. We suggest one approach to each step here. If you assume that ξ is continuous then one approach to the first step is to show that the function G above can be expressed as

$$G(w,s) = \frac{2^D}{|\mathbb{S}^{D-1}|} \iint \delta(w - v' - v'_*)\, \delta(s - |v'|^2 + |v'_*|^2)\, \frac{\xi' + \xi'_*}{|v'_* - v'|^{D-2}} \, \mathrm{d}v'_* \, \mathrm{d}v' \,.$$

If you assume that ξ is twice differentiable (so that G is too) then one approach to the second step is to show that

$$\nabla_w \nabla_w G(w,s) = 0 \,, \qquad \nabla_w \partial_s G(w,s) = 0 \,, \qquad \partial_{ss} G(w,s) = 0 \,,$$

from which it follows that G is a linear function of w and s. When ξ is only locally integrable one can first regularize it by convolution, then use the above arguments, and finally remove the regularization. A proof of (3.12) that assumes ξ is merely measurable and finite almost everywhere can be found in [3]. □

3.4. **Conservation Laws.** Locally conserved quantities lead to local conservation laws. If $\xi = \xi(v)$ is a locally conserved quantity and F solves the Boltzmann equation (**??**) then by (3.12b) we obtain the local conservation laws

$$\partial_t \langle \xi\, F \rangle + \nabla_x \cdot \langle v\, \xi\, F \rangle = 0 \,. \tag{3.15}$$

By characterization (3.13) the locally conserved quantities are $\mathrm{span}\{1, v_{_1}, v_{_2}, \cdots, v_{_D}, |v|^2\}$. By successively setting $\xi = 1, v, \frac{1}{2}|v|^2$ in (3.15), we obtain the local conservation laws of mass, momentum, and energy:

$$\partial_t \langle F \rangle + \nabla_x \cdot \langle v\, F \rangle = 0\,, \tag{3.16a}$$

$$\partial_t \langle v\, F \rangle + \nabla_x \cdot \langle v \otimes v\, F \rangle = 0\,, \tag{3.16b}$$

$$\partial_t \langle \tfrac{1}{2}|v|^2 F \rangle + \nabla_x \cdot \langle v \tfrac{1}{2}|v|^2 F \rangle = 0\,. \tag{3.16c}$$

Every local conservation law of the form (3.15) for some $\xi = \xi(v)$ is a linear combination of these.

When there is no contribution from the boundary terms, the local conservation laws (3.16) can be integrated over space and time to obtain global conservation laws of mass, momentum, and energy:

$$\begin{aligned} \int \langle F(t) \rangle \,\mathrm{d}x &= \int \langle F^{in} \rangle \,\mathrm{d}x\,, \\ \int \langle v\, F(t) \rangle \,\mathrm{d}x &= \int \langle v\, F^{in} \rangle \,\mathrm{d}x\,, \\ \int \langle \tfrac{1}{2}|v|^2 F(t) \rangle \,\mathrm{d}x &= \int \langle \tfrac{1}{2}|v|^2 F^{in} \rangle \,\mathrm{d}x\,. \end{aligned} \tag{3.17}$$

In particular, these imply that if the mass and energy of F are finite initially then the mass, momentum and energy of F are finite for all time.

Other local conservation laws can be derived from those in (3.16). For example, taking the tensor product of the momentum law (3.16b) with x and subtracting its transpose yields

$$\partial_t \langle (v \otimes x - x \otimes v)\, F \rangle + \nabla_x \cdot \langle v \otimes (v \otimes x - x \otimes v)\, F \rangle = 0\,. \tag{3.18}$$

This is the local conservation law of angular momentum. Next, multiplying the mass law (3.16a) by x, multiplying the momentum law (3.16b) by t, and subtracting the results yields

$$\partial_t \langle (x - vt)\, F \rangle + \nabla_x \cdot \langle v\, (x - vt)\, F \rangle = 0\,. \tag{3.19}$$

Similarly, taking the dot product of the momentum law (3.16b) with x and multiplying the energy law (3.16c) by $2t$ and subtracting yields

$$\partial_t \langle v \cdot (x - vt)\, F \rangle + \nabla_x \cdot \langle v\, v \cdot (x - vt)\, F \rangle = 0\,. \tag{3.20}$$

Finally, multiplying the mass law (3.16a) by $\frac{1}{2}|x|^2$, taking the dot product of the momentum law (3.16b) with $-xt$, multiplying the energy law (3.16c) by $\frac{1}{2}t^2$, and adding the results yields

$$\partial_t \langle \tfrac{1}{2}|x - vt|^2 F \rangle + \nabla_x \cdot \langle v \tfrac{1}{2}|x - vt|^2 F \rangle = 0\,. \tag{3.21}$$

When there is no contribution from the boundary terms, integrating the local conservation laws (3.18–3.21) over space and time yields

$$
\begin{aligned}
\int \langle (v \otimes x - x \otimes v)\, F(t) \rangle \,\mathrm{d}x &= \int \langle (v \otimes x - x \otimes v)\, F^{in} \rangle \,\mathrm{d}x \,, \\
\int \langle (x - vt)\, F(t) \rangle \,\mathrm{d}x &= \int \langle x\, F^{in} \rangle \,\mathrm{d}x \,, \\
\int \langle v \cdot (x - vt)\, F(t) \rangle \,\mathrm{d}x &= \int \langle v \cdot x\, F^{in} \rangle \,\mathrm{d}x \,, \\
\int \langle \tfrac{1}{2}|x - vt|^2 F(t) \rangle \,\mathrm{d}x &= \int \langle \tfrac{1}{2}|x|^2 F^{in} \rangle \,\mathrm{d}x \,.
\end{aligned} \tag{3.22}
$$

The first is the global conservation law of angular momentum. The second governs the dynamics of the center of mass. The last two are related to the dynamics of the moment of inertia. When the last three of these are conbined with the global conservation laws of momentum, and energy we obtain

$$
\begin{aligned}
\int \langle x\, F(t) \rangle \,\mathrm{d}x &= \int \langle (x + vt)\, F^{in} \rangle \,\mathrm{d}x \,, \\
\int \langle v \cdot x\, F(t) \rangle \,\mathrm{d}x &= \int \langle v \cdot (x + vt)\, F^{in} \rangle \,\mathrm{d}x \,, \\
\int \langle \tfrac{1}{2}|x|^2 F(t) \rangle \,\mathrm{d}x &= \int \langle \tfrac{1}{2}|x + vt|^2 F^{in} \rangle \,\mathrm{d}x \,.
\end{aligned} \tag{3.23}
$$

In particular, these imply that if the mass, energy, and moment of inertia of F are finite initially then the mass, momentum, energy, angular momentum, center of mass, mixed moment, and moment of inertia of F are finite for all time.

The local conservation laws (3.18–3.21) each have the form (3.15) where

$$
\xi = a(x,t) + v \cdot b(x,t) + \tfrac{1}{2}|v|^2 c(x,t) \,, \tag{3.24a}
$$

with

$$
\partial_t \xi + v \cdot \nabla_x \xi = 0 \,. \tag{3.24b}
$$

Conversely, if $\xi = \xi(v,x,t)$ is continuously differentiable and satisfies (3.24b) while F solves the Boltzmann equation (**??**) then

$$
\partial_t(\xi\, F) + \nabla_x \cdot (v\, \xi\, F) = \xi\, \mathcal{B}(F,F) \,,
$$

which when integrated yields (3.15). The following lemma, which is essentially due to Boltzmann for the case $D = 3$, shows that every local conservation law of this type is a linear combination of those in (3.16) and (3.18–3.21).

Lemma. If $\xi = \xi(v,x,t)$ is continuously differentiable and satisfies (3.24) then

$$
\begin{aligned}
\xi = a_0 + v \cdot b_0 + \tfrac{1}{2}|v|^2 c_0 + v \cdot B_1 x + (x - vt) \cdot b_1 \\
+ v \cdot (x - vt) c_1 + \tfrac{1}{2}|x - vt|^2 c_2 \,,
\end{aligned} \tag{3.25}
$$

for some a_0, c_0, c_0, $c_0 \in \mathbb{R}$, b_0, $b_1 \in \mathbb{R}^D$, and $B_1 \in \mathbb{R}^{D \vee D}$ such that $B_1^T = -B_1$.

Proof. Because

$$
\begin{aligned}
\partial_t \xi + v \cdot \nabla_x \xi = \partial_t a + v \cdot \nabla_x a + v \cdot \partial_t b + v \otimes v : \nabla_x b \\
+ \tfrac{1}{2}|v|^2 \partial_t c + \tfrac{1}{2}|v|^2 v \cdot \nabla_x c \,,
\end{aligned}
$$

it is clear that $\partial_t \xi + v \cdot \nabla_x \xi = 0$ if and only if

$$\partial_t a = 0\,, \qquad \partial_t b + \nabla_x a = 0\,, \qquad \partial_t c + \tfrac{2}{D}\nabla_x \cdot b = 0\,,$$
$$\nabla_x b + (\nabla_x b)^T - \tfrac{2}{D}\nabla_x \cdot bI = 0\,, \qquad \nabla_x c = 0\,.$$

The facts $\partial_t a = 0$ and $\partial_t b + \nabla_x a = 0$ imply that $\partial_{tt} b = 0$. The facts $\partial_{tt} b = 0$ and $\partial_t c + \frac{2}{D}\nabla_x \cdot b = 0$ imply that $\partial_{ttt} c = 0$. The facts $\partial_{ttt} c = 0$ and $\nabla_x c = 0$ imply that

$$c = c_0 - 2c_1 t + c_2 t^2\,, \quad \text{where } c_0,\, c_1,\, c_2 \in \mathbb{R}\,.$$

This fact plus the facts $\partial_t c + \frac{2}{D}\nabla_x \cdot b = 0$ and $\nabla_x b + (\nabla_x b)^T - \frac{2}{D}\nabla_x \cdot bI = 0$ imply that

$$\nabla_x b + (\nabla_x b)^T = \tfrac{2}{D}\nabla_x \cdot bI = -\partial_t c I = (2c_1 - 2c_2 t)I\,.$$

This fact plus the fact that $\partial_{tt} b = 0$ implies that

$$b = b_0 - b_1 t + B_1 x - B_2 x t + c_1 x - c_2 x t\,, \quad \text{where } B_1^T = -B_1 \text{ and } B_2^T = -B_2\,.$$

This fact plus the fact that $\partial_t b + \nabla_x a = 0$ imply that

$$\nabla_x a = -\partial_t b = b_1 + B_2 x + c_2 x\,.$$

This fact plus the fact that $\partial_t a = 0$ implies that $B_2 = 0$ and

$$a = a_0 + b_1 \cdot x + \tfrac{1}{2} c_2 |x|^2\,, \quad \text{where } a_0 \in \mathbb{R}\,.$$

Therefore ξ has the form (3.25). □

3.5. **Entropy, Dissipation, and Local Equilibria.** Upon setting $\xi = \log(f)$ in identity (3.11), Boltzmann obtained

$$\langle \log(f)\, \mathcal{B}(f,f)\rangle = \tfrac{1}{4} \iiint \big(\log(f) + \log(f_*) - \log(f') - \log(f'_*)\big) \\ (f'_* f' - f_* f)\, b \, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v\,,$$

for every $f = f(v)$ for which the integrals make sense. By using the additive properties of the logarithm he observed that the above integrand is non-positive almost everywhere, and thereby obtained the dissipation law

$$\langle \log(f)\, \mathcal{B}(f,f)\rangle = -\tfrac{1}{4} \iiint \log\!\left(\frac{f'_* f'}{f_* f}\right) (f'_* f' - f_* f)\, b \, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v \le 0\,. \tag{3.26}$$

He then characterized the equilibria of the collision operator; he found that for any $f = f(v)$ for which the integrals make sense, the following statements are equivalent:

(3.27)

(i) $\langle \log(f)\, \mathcal{B}(f,f)\rangle = 0\,;$

(ii) $f'_* f' = f_* f$ almost everywhere;

(iii) $\mathcal{B}(f,f) = 0\,;$

(iv) $\log(f) \in \mathrm{span}\{1, v_1, v_2, \cdots, v_D, |v|^2\}\,.$

Properties (3.26) and (3.27) are the key results of Boltzmann's celebrated H-theorem. The implication (i) $\Longrightarrow$ (ii) follows from the dissipation law (3.26). The implication (ii) $\Longrightarrow$ (iii) follows from formula (**??**) for $\mathcal{B}$. The implication (iii) $\Longrightarrow$ (i) is obvious. The implication (iv) $\Longrightarrow$ (i) follows from the conservation laws (3.13). Finally, the implication (ii) $\Longrightarrow$ (iv) follows upon taking the log of (ii) to see that

$$\log(f'_*) + \log(f') = \log(f_*) + \log(f) \quad \text{almost everywhere}\,,$$

thereby seeing that $\log(f)$ is a collision invariant, which by the characterization of collision invariants (3.12) implies (iv).

The equilibria characterized in (iv) above that have finite mass, momentum, and energy density are the classical Maxwellians. They have the form $f = \mathcal{M}(\rho, u, \theta)$ where $\mathcal{M}(\rho, u, \theta)$ is given by

$$\mathcal{M}(v; \rho, u, \theta) \equiv \frac{\rho}{(2\pi\theta)^{D/2}} \exp\left(-\frac{|v-u|^2}{2\theta} \right), \tag{3.28}$$

for some $\rho \geq 0$, $u \in \mathbb{R}^D$, and $\theta > 0$. The values of these parameters are determined by f through the relations

$$\rho = \langle f \rangle, \qquad \rho u = \langle v\, f \rangle, \qquad \tfrac{1}{2}\rho|u|^2 + \tfrac{D}{2}\rho\,\theta = \langle \tfrac{1}{2}|v|^2 f \rangle. \tag{3.29}$$

One sees that ρ is the mass density, u is the bulk velocity, and θ is related to the temperature T by $\theta = kT/m$ where k is the Boltzmann constant and m is the molecule mass.

Now, if F solves the Boltzmann equation (**??**) then the dissipation law (3.26) implies that F satisfies the local entropy dissipation law

$$\begin{aligned} &\partial_t \langle F\log(F) - F\rangle + \nabla_x \cdot \langle v\,(F\log(F) - F)\rangle \\ &-\tfrac{1}{4} \iiint \log\left(\frac{F'_*F'}{F_*F}\right) (F'_*F' - F_*F)\, b\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v \leq 0\,. \end{aligned} \tag{3.30}$$

When there is no contribution from the boundary terms, integrating this over space and time yields the global entropy equality

$$H(F(t)) + \int_0^t R(F(s))\,\mathrm{d}s = H(F^{in}), \tag{3.31}$$

where $H(F)$ is the entropy functional

$$H(F) = \int \langle F\log(F) - F\rangle\, \mathrm{d}x\,, \tag{3.32}$$

and $R(F)$ is the entropy dissipation rate functional

$$R(F) = \frac{1}{4} \iiiint \log\left(\frac{F'_*F'}{F_*F}\right) (F'_*F' - F_*F)\, b\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v\, \mathrm{d}x\,. \tag{3.33}$$

Here we adopt the sign convention of diminishing entropy which, while at variance with much of the physics literature, is in many ways more natural.

3.6. **Global Maxwellians.** It is natural to ask which local Maxwellians are solutions of the Boltzmann equation. In other words, to ask for which $\rho(x,t)$, $u(x,t)$, and $\theta(x,t)$ does $\mathcal{M}(v; \rho(x,t), u(x,t), \theta(x,t))$ solve the Boltzmann equation. Such a local Maxwellian is called a *global Maxwellian*. Because local Maxwellians annihilate the collision operator, a continuously differentiable local Maxwellian $\mathcal{M}(v; \rho, u, \theta)$ is a global Maxwellian if and only if

$$\big(\partial_t + v\cdot\nabla_x\big)\mathcal{M}(v; \rho, u, \theta) = 0\,.$$

Of course this will be satisfied when ρ, u, and θ are constants. These are the so-called *homogeneous Maxwellians*. However, as the following result shows there are other other global Maxwellians.

Theorem 3.1. *The function $F(v,x,t)=\mathcal{M}(v;\rho(x,t),u(x,t),\theta(x,t))$ is a global (in time) thrice differentiable solution to the Boltzmann equation if and only if*

(3.34)
$$\rho(x,t)=\left(\frac{2\pi}{at^2-2bt+c}\right)^{\frac{D}{2}}\exp\!\left(-\left(\tfrac{1}{2}a|x|^2-f\cdot x+h-\frac{|axt-bx-Bx-ft+g|^2}{2(at^2-2bt+c)}\right)\right),$$
$$u(x,t)=\frac{axt-bx-Bx-ft+g}{at^2-2bt+c}\,,\qquad \theta(x,t)=\frac{1}{at^2-2bt+c}\,.$$

where $a,b,c,h\in\mathbb{R}$, $f,g\in\mathbb{R}^D$, and $B\in\mathbb{R}^{D\vee D}$ such that $c>0$, $ac>b^2$, and $B^T=-B$. This function is moreover integrable over $\mathbb{R}^D$ if and only if the symmetric matrix

$$(ac-b^2)I+B^2 \qquad \textit{is also positive definite}. \tag{3.35}$$

Remark. This result was proved by Boltzmann for the case $D=3$. His classic text even treats a setting with external forces.

Proof. Formula (3.26) shows that $\log(\mathcal{M})$ has the form

$$-\log(\mathcal{M}(v;\rho(x,t),u(x,t),\theta(x,t))=a(x,t)+v\cdot b(x,t)+\tfrac{1}{2}|v|^2c(x.t)\,,$$

where a, b, and c are thrice differentiable and $c(x,t)>0$ for every (x,t). Set $a^{in}(x)=a(x,0)$, $b^{in}(x)=b(x,0)$, and $c^{in}(x)=c(x,0)$. Because

$$(\partial_t+v\cdot\nabla_x)\log(\mathcal{M})=0\,,$$

we see that $\log(\mathcal{M})$ also has the form

$$-\log(\mathcal{M}(v;\rho(x,t),u(x,t),\theta(x,t))=a^{in}(x-vt)+v\cdot b^{in}(x-vt)+\tfrac{1}{2}|v|^2c^{in}(x-vt)\,,$$

By equating these two expressions for $\log(\mathcal{M})$, we find that

$$a(x,t)+v\cdot b(x,t)+\tfrac{1}{2}|v|^2c(x.t)=a^{in}(x-vt)+v\cdot b^{in}(x-vt)+\tfrac{1}{2}|v|^2c^{in}(x-vt)\,. \tag{3.36}$$

By differentiating this relation three times with respect to v, we obtain

$$\begin{aligned}
b(x,t)+vc(x,t)&=b^{in}(x-vt)+vc^{in}(x-vt)\\
&\quad-t\nabla_x a^{in}(x-vt)-t\nabla_x\big(v\cdot b^{in}(x-vt)\big)-t\nabla_x\big(\tfrac{1}{2}|v|^2c^{in}(x-vt)\big)\,,\\
Ic(x,t)&=Ic^{in}(x-vt)-2t\nabla_x\vee b^{in}(x-vt)-2tv\vee\nabla_x c^{in}(x-vt)\\
&\quad+t^2\nabla_x^2a^{in}(x-vt)+t^2\nabla_x^2\big(v\cdot b^{in}(x-vt)\big)+t^2\nabla_x^2\big(\tfrac{1}{2}|v|^2c^{in}(x-vt)\big)\,,\\
0&=-3tI\vee\nabla_x c^{in}(x-vt)+3t^2\nabla_x^2\vee b^{in}(x-vt)+3t^2v\vee\nabla_x^2c^{in}(x-vt)\\
&\quad-t^3\nabla_x^3a^{in}(x-vt)-t^3\nabla_x^3\big(v\cdot b^{in}(x-vt)\big)-t^3\nabla_x^3\big(\tfrac{1}{2}|v|^2c^{in}(x-vt)\big)\,.
\end{aligned} \tag{3.37}$$

Upon dividing the last equation by t and then setting $t=0$, we see that $I\vee\nabla_x c^{in}(x)=0$, which implies $\nabla_x c^{in}(x)=0$. The last equation thereby reduces to

$$0=3\nabla_x^2\vee b^{in}(x-vt)-t\nabla_x^3a^{in}(x-vt)-t\nabla_x^3\big(v\cdot b^{in}(x-vt)\big)\,.$$

Upon setting $t=0$ in this equation, we see that $\nabla_x^2\vee b^{in}(x)=0$, whereby the above equation reduces to

$$=\nabla_x^3a^{in}(x-vt)+\nabla_x^3\big(v\cdot b^{in}(x-vt)\big)\,.$$

Upon setting $t=0$ in this equation and using the linear independence of 1 and the components of v, we see that $\nabla_x^3a^{in}(x)=0$ and $\nabla_x^3b^{in}(x)=0$.

The above analysis shows that the last equation in (3.36) is satisfied if and only if

$$\nabla_x c^{in}(x) = 0\,, \qquad \nabla_x^2 \vee b^{in}(x) = 0\,,$$
$$\nabla_x^3 a^{in}(x) = 0\,, \qquad \nabla_x^3 b^{in}(x) = 0\,.$$

These equations hold if and only if

$$\begin{aligned} a^{in}(x) &= \tfrac{1}{2} a^{(2)}_{ij} x_i x_j + a^{(1)}_j x_j + a^{(0)}\,,\\ b^{in}_k(x) &= \tfrac{1}{2} b^{(2)}_{ijk} x_i x_j + b^{(1)}_{jk} x_j + b^{(0)}_k\,,\\ c^{in}(x) &= c^{(0)}\,, \end{aligned}$$

where

$$a^{(2)}_{ij} = a^{(2)}_{ji}\,, \qquad b^{(2)}_{ijk} = b^{(2)}_{jik}\,, \qquad c^{(0)} > 0\,,$$
$$b^{(2)}_{ijk} + b^{(2)}_{jki} + b^{(2)}_{kij} = 0\,.$$

When the above formulas for $a^{in}(x)$, $b^{in}(x)$, and $c^{in}(x)$ are placed into the second equation of (3.36) and we use the last relation above (cyclic symmetry for $b^{(2)}_{ijk}$), we obtain

$$\begin{aligned} \delta_{ij} c(x,t) &= \delta_{ij} c^{(0)} - t\Big(b^{(2)}_{kij} + b^{(2)}_{jki}\Big)(x_k - v_k t) - t\Big(b^{(1)}_{ij} + b^{(1)}_{ji}\Big)\\ &\qquad + t^2 a^{(2)}_{ij} + t^2 b^{(2)}_{ijk} v_k\\ &= \delta_{ij} c^{(0)} + t b^{(2)}_{ijk} x_k - t\Big(b^{(1)}_{ij} + b^{(1)}_{ji}\Big) + t^2 a^{(2)}_{ij}\,. \end{aligned} \tag{3.38}$$

The trace of this equation yields

$$D c(x,t) = D c^{(0)} + t b^{(2)}_{llk} x_k - 2t b^{(1)}_{ll} + t^2 a^{(2)}_{ll}\,,$$

which can then be used to eliminate $c(x,t)$ and $c^{(0)}$ in (3.38). We thereby obtain

$$\begin{aligned} 0 = &\Big(b^{(2)}_{ijk} - \tfrac{1}{D} b^{(2)}_{llk} \delta_{ij}\Big) x_k\\ &- \Big(b^{(1)}_{ij} + b^{(1)}_{ji} - \tfrac{2}{D} b^{(1)}_{ll} \delta_{ij}\Big)\\ &+ t\Big(a^{(2)}_{ij} - \tfrac{1}{D} a^{(2)}_{ll} \delta_{ij}\Big)\,. \end{aligned}$$

But this holds if and only if

$$a^{(2)}_{ij} = \tfrac{1}{D} a^{(2)}_{ll} \delta_{ij}\,, \qquad b^{(2)}_{ijk} = \tfrac{1}{D} b^{(2)}_{llk} \delta_{ij}\,,$$
$$b^{(1)}_{ij} + b^{(1)}_{ji} - \tfrac{2}{D} b^{(1)}_{ll} \delta_{ij} = 0\,.$$

By again using the cyclic symmetry for $b^{(2)}_{ijk}$, we see that $b^{(2)}_{ijk} = \frac{1}{D} b^{(2)}_{llk} \delta_{ij}$ implies that $b^{(2)}_{ijk} = 0$.

Now define $a, b, c, h \in \mathbb{R}$, $f, g \in \mathbb{R}^D$, and $B \in \mathbb{R}^{D \vee D}$ by

$$a = \tfrac{1}{D} a^{(2)}_{ll}\,, \qquad b = \tfrac{1}{D} b^{(1)}_{ll}\,, \qquad c = c^{(0)}\,, \qquad h = a^{(0)}\,,$$
$$f_j = -a^{(1)}_j\,, \qquad g_k = -b^{(0)}_k\,, \qquad B_{ij} = b^{(1)}_{ij} - \tfrac{1}{D} b^{(1)}_{ll} \delta_{ij}\,.$$

Then

$$a^{in}(x) = \tfrac{1}{2} a|x|^2 - f \cdot x + h\,, b^{in}(x) \;\; = bx + Bx - g\,, c^{in}(x) = c\,, \qquad a(x,t) = \tfrac{1}{2} a|x|^2 - f \cdot x + h\,, b(x,t) \;\; = b$$

It is easily checked that

References

[1] L. Boltzmann, *Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen*, Sitzungs. Akad. Wiss. Wein **66** (1872), 275–370; English: *Further Studies on the Thermal Equilibrium of Gas Molecules*, in *Kinetic Theory* **2**, S.G. Brush (ed.), Pergamon Press, London, 1966, 88–174.

[2] L. Boltzmann, *Vorlesungen über Gastheorie*, J.A. Barth, Leipzig, 1896 (Part I) & 1898 (Part II) English: *Lectures on Gas Theory*, translated by S.G. Brush, Dover, New York, 1995.

[3] C. Cercignani, R. Illner, and M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Appl. Math. Sci. **106**, Springer-Verlag, New York, 1994.