Quiz 8 Solutions, Math 246, Professor David Levermore
Tuesday, 12 April 2011

Short Table: $\mathcal{L}[\sin(bt)](s) = \dfrac{b}{s^2+b^2}$ for $s > 0$, $\quad \mathcal{L}[e^{at}](s) = \dfrac{1}{s-a}$ for $s > a$.

(1) [3] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for $f(t) = u(t-2)$, where u is the unit step function.

Solution. By the definitions of the Laplace transform and the unit step function

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{-st} u(t-2)\,\mathrm{d}t = \lim_{T\to\infty} \int_2^T e^{-st}\,\mathrm{d}t\,.$$

The above limit diverges for $s \leq 0$. For $s > 0$

$$\int_2^T e^{-st}\,\mathrm{d}t = -\frac{e^{-st}}{s}\bigg|_2^T = \frac{e^{-s2}}{s} - \frac{e^{-sT}}{s}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[\frac{e^{-s2}}{s} - \frac{e^{-sT}}{s}\right] = \frac{e^{-2s}}{s} \quad \text{for } s > 0\,.$$

(2) [4] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$y'' + 9y = \sin(2t)\,, \quad y(0) = 0\,, \quad y'(0) = 5\,.$ DO NOT solve for $y(t)$, just $Y(s)$!

Solution. The Laplace transform of the initial-value problem and item 1 in the table at the top of the page with $b = 2$ gives

$$\mathcal{L}[y''](s) + 9\mathcal{L}[y](s) = \mathcal{L}[\sin(2t)](s) = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}\,,$$

where

$$\mathcal{L}[y](s) = Y(s)\,,$$
$$\mathcal{L}[y''](s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 5\,.$$

Hence,

$$(s^2+9)Y(s) = 5 + \frac{2}{s^2+4}\,, \quad \implies \quad Y(s) = \frac{5}{s^2+9} + \frac{2}{(s^2+9)(s^2+4)}\,.$$

(3) [3] Find the inverse Laplace transform $y(t)$ of the function $Y(s) = \dfrac{s+1}{s^2-10s+21}$.

Solution. By partial fractions

$$Y(s) = \frac{s+1}{s^2-10s+21} = \frac{s+1}{(s-3)(s-7)} = \frac{-1}{s-3} + \frac{2}{s-7}\,.$$

Item 2 in the table at the top of the page with $a = 3$ and with $a = 7$ then gives

$$y(t) = \mathcal{L}^{-1}[Y](t) = \mathcal{L}^{-1}\left[\frac{-1}{s-3} + \frac{2}{s-7}\right]$$
$$= -\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s-7}\right] = -e^{3t} + 2e^{7t}\,.$$