Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 1 February 2011

(1) [4] For each of the following ordinary differential equations, give its order and state whether it is linear or nonlinear.

(a) $\dfrac{\mathrm{d}^3 w}{\mathrm{d}x^3} + w^2 \dfrac{\mathrm{d}w}{\mathrm{d}x} + e^x w = x^2$; **Solution.** third order, nonlinear.

(b) $\dfrac{\mathrm{d}^5 y}{\mathrm{d}s^5} = 2s\,\dfrac{\mathrm{d}^2 y}{\mathrm{d}s^2} + \sin(s)$. **Solution.** fifth order, linear.

(2) [4] Solve the initial-value problem

$$t\frac{\mathrm{d}z}{\mathrm{d}t} = 3z + 2t\,, \qquad z(1) = 0\,.$$

Solution. This equation is linear. Its normal form is

$$\frac{\mathrm{d}z}{\mathrm{d}t} - \frac{3}{t}\,z = 2\,.$$

An integrating factor is $e^{A(t)}$ where $A'(t) = -3/t$. Setting $A(t) = -3\log(t)$, we find that $e^{A(t)} = e^{-3\log(t)} = t^{-3}$. Hence, the problem has the intgrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t^{-3}z\big) = t^{-3}\cdot 2 = 2t^{-3}\,.$$

Integrating both sides yields

$$t^{-3}z = -t^{-2} + c\,.$$

Imposing the initial condition gives

$$1^{-3}\cdot 0 = -1^{-2} + c\,,$$

whereby $c = 1^{-2} = 1$. The solution is therefore

$$z = -t + t^3\,, \qquad \text{for every } t > 0\,.$$

Remark. The interval of definition for this solution is $(0,\infty)$. Can you see why?

(3) [2] Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{t^2-1}\,x = \frac{1}{\sin(t)}\,, \qquad x(2) = -3\,.$$

(You do not have to solve this equation to answer this question!)

Solution. This equation is linear and is already in normal form. The coefficient $1/(t^2-1)$ is continuous everywhere except at $t = \pm 1$, while the forcing $1/\sin(t)$ is continuous everywhere except where $t = n\pi$ for some integer n. You can therefore read off that the interval of definition for its solution is $(1,\pi)$ because:

- the initial time $t = 2$ is in $(1,\pi)$,
- the coefficient and forcing are both continuous over $(1,\pi)$,
- the coefficient is not defined at $t = 1$,
- the forcing is not defined at $t = \pi$.