Solutions to First In-Class Exam: Math 401
Section 0201, Professor Levermore
Monday, 8 March 2010

1. [15] The coefficient matrix A of the 4×5 linear system $A\mathbf{x} = \mathbf{f}$ can be row reduced to

$$\begin{pmatrix} 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

(a) What is the rank of A?
(b) Does there exist a solution of $A\mathbf{x} = \mathbf{f}$ for every $\mathbf{f} \in \mathbb{R}^4$? If not, how many solvability conditions must $\mathbf{f}$ satisfy?
(c) If a solition of $A\mathbf{x} = \mathbf{f}$ exists for some $\mathbf{f} \in \mathbb{R}^4$, is it unique? If not, how many free parameters does a general solution have?
(d) Give a general solution of $A\mathbf{x} = \mathbf{0}$.

Solution (a). Because the reduced matrix has 3 pivots, $\text{rank}(A) = 3$.

Solution (b). Because A has 4 rows and $\text{rank}(A) = 3$, $\mathbf{f}$ must satisfy $4 - 3 = 1$ solvability condition. In particular, the system $A\mathbf{x} = \mathbf{f}$ does not have a solution for every $\mathbf{f} \in \mathbb{R}^4$.

Solution (c). Because A has 5 columns and $\text{rank}(A) = 3$, any general solution will have $5 - 3 = 2$ free parameters. In particular, if a solution of the system $A\mathbf{x} = \mathbf{f}$ exists for some $\mathbf{f} \in \mathbb{R}^4$, it will not be unique.

Solution (d). A general solution of $A\mathbf{x} = \mathbf{0}$ with free parameters c_1 and c_2 is

$$x_1 = -3c_1 + c_2 , \quad x_2 = c_1 , \quad x_3 = -4c_2 , \quad x_4 = 2c_2 , \quad x_5 = c_2 .$$

2. [15] Let $A = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 3 \\ 2 & 7 & 8 \end{pmatrix}$. Compute (a) $\det(A)$, (b) $\text{Cof}(A)$, (c) A^{-1}.

Solution (a). The "signed diagonal products" formula for 3×3 determinants yields

$$\begin{aligned} \det(A) = \det \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 3 \\ 2 & 7 & 8 \end{pmatrix} &= 1 \cdot 1 \cdot 8 + 1 \cdot 3 \cdot 2 + (-1) \cdot (-2) \cdot 7 \\ &\quad - 2 \cdot 1 \cdot (-1) - 7 \cdot 3 \cdot 1 - 8 \cdot (-2) \cdot 1 \\ &= 8 + 6 + 14 + 2 - 21 + 16 = 25 . \end{aligned}$$

Alternate Solution (a). A Laplace expansion along the top row yields

$$\begin{aligned} \det(A) = \det \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 3 \\ 2 & 7 & 8 \end{pmatrix} &= \det \begin{pmatrix} 1 & 3 \\ 7 & 8 \end{pmatrix} - \det \begin{pmatrix} -2 & 3 \\ 2 & 8 \end{pmatrix} - \det \begin{pmatrix} -2 & 1 \\ 2 & 7 \end{pmatrix} \\ &= -13 + 22 + 16 = 25 . \end{aligned}$$

Second Alternate Solution (a). Row reduction shows that

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 3 \\ 2 & 7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 5 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{25}{3} \end{pmatrix} .$$

The product of the pivots then yields $\det(A) = 1 \cdot 3 \cdot \frac{25}{3} = 25$.

Solution (b). The cofactor matrix formula yields

$$\mathrm{Cof}(A) = \begin{pmatrix} \det\begin{pmatrix} 1 & 3 \\ 7 & 8 \end{pmatrix} & -\det\begin{pmatrix} -2 & 3 \\ 2 & 8 \end{pmatrix} & \det\begin{pmatrix} -2 & 1 \\ 2 & 7 \end{pmatrix} \\ -\det\begin{pmatrix} 1 & -1 \\ 7 & 8 \end{pmatrix} & \det\begin{pmatrix} 1 & -1 \\ 2 & 8 \end{pmatrix} & -\det\begin{pmatrix} 1 & 1 \\ 2 & 7 \end{pmatrix} \\ \det\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} & -\det\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -13 & 22 & -16 \\ -15 & 10 & -5 \\ 4 & -1 & 3 \end{pmatrix} .$$

Solution (c). We can combine the solutions to parts (a) and (b) to obtain

$$A^{-1} = \frac{1}{\det(A)} \mathrm{Cof}(A)^T = \frac{1}{25} \begin{pmatrix} -13 & -15 & 4 \\ 22 & 10 & -1 \\ -16 & -5 & 3 \end{pmatrix} .$$

Alternative Solution (c). By row reduction

$$\begin{pmatrix} A & | & I \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ 2 & 7 & 8 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 2 & 1 & 0 \\ 0 & 5 & 10 & -2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 5 & 10 & -2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{4}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{25}{3} & -\frac{16}{3} & -\frac{5}{3} & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{4}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{16}{25} & -\frac{1}{5} & \frac{3}{25} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{13}{25} & -\frac{3}{5} & \frac{4}{25} \\ 0 & 1 & 0 & \frac{22}{25} & \frac{2}{5} & -\frac{1}{25} \\ 0 & 0 & 1 & -\frac{16}{25} & -\frac{1}{5} & \frac{3}{25} \end{array}\right) .$$

We can then read off that

$$A^{-1} = \begin{pmatrix} -\frac{13}{25} & -\frac{3}{5} & \frac{4}{25} \\ \frac{22}{25} & \frac{2}{5} & -\frac{1}{25} \\ -\frac{16}{25} & -\frac{1}{5} & \frac{3}{25} \end{pmatrix} .$$

3. [20] Consider the linear system

$$\begin{pmatrix} 1 & 1 & b \\ b & 3 & -1 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ c \end{pmatrix} .$$

For what real values of b and c does this system (a) have a unique solution, (b) have no solution, (c) have many solutions?

Solution (a). Row reduction of the augmented matrix yields

$$\left(\begin{array}{ccc|c} 1 & 1 & b & 1 \\ b & 3 & -1 & -2 \\ 3 & 4 & 1 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & b & 1 \\ 0 & 3-b & -1-b^2 & -2-b \\ 0 & 1 & 1-3b & c-3 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 1 & b & 1 \\ 0 & 1 & 1-3b & c-3 \\ 0 & 3-b & -1-b^2 & -2-b \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 1 & b & 1 \\ 0 & 1 & 1-3b & c-3 \\ 0 & 0 & -4+10b-4b^2 & bc-4b-3c+7 \end{array}\right) .$$

The system will have a unique solution if and only if $-4+10b-4b^2 \neq 0$. Upon factoring we see that

$$-4b^2 + 10b - 4 = -2(b-2)(2b-1) .$$

Therefore the system will have a unique solution if and only if $b \neq 2$ and $b \neq \frac{1}{2}$.

Solution (b). When $b = 2$ or $b = \frac{1}{2}$ the system has the solvability condition $(b-3)c = 4b-7$. Therefore the system will have no solution if and only if either $b = 2$ and $c \neq -1$, or $b = \frac{1}{2}$ and $c \neq 2$.

Solution (c). When $b = 2$ or $b = \frac{1}{2}$ the system has the solvability condition $(b-3)c = 4b-7$. Therefore the system will have many solutions if and only if either $b = 2$ and $c = -1$, or $b = \frac{1}{2}$ and $c = 2$.

4. [20] Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -2 \\ 2 & -1 & 2 \\ 1 & -1 & 3 \end{pmatrix} .$$

(a) Give a test to show that A has an LU factorization.
(b) Find the LU factorization of A.
(c) Find the LDU factorization of A.
(d) Find $\det(A)$.

Solution (a). An $m \times m$ matrix will have an LU factorization if the determinants of its first $m-1$ principle minors are nonzero. For the matrix A these determiants are

$$\det(1) = 1 , \qquad \det\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = -1 - (-2) = 1 ,$$

so that A has an LU factorization.

Solution (b). We row reduce A into upper upper triangular form by

$$A = \begin{pmatrix} 1 & -1 & -2 \\ 2 & -1 & 2 \\ 1 & -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 6 \\ 0 & 0 & 5 \end{pmatrix} = U\,,$$

where we have multiplied the first row by 2 and subtracted it from the second, and have multiplied the first row by 1 and subtracted it from the third. These row operations are encoded by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

Therefore A has the LU factorization

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 6 \\ 0 & 0 & 5 \end{pmatrix} .$$

Solution (c). It follows from (b) that A has the LDU factorization

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 6 \\ 0 & 0 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} . \end{aligned}$$

Solution (d). Because the only row operations used in (b) where adding multiples of one row to another, it follows that that $\det(A) = \det(U) = 1 \cdot 1 \cdot 5 = 5$.

5. [15] Give the 4×4 elementary matrices corresponding to the following elementary row operations:
   (a) multiplying row 3 by -5,
   (b) exchanging rows 2 and 4,
   (c) adding 7 times row 4 to row 1.

Solution (a). $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

Solution (b). $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$

Solution (c). $\begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

6. [15] Compute A^{-1} for

$$\text{(a)} \quad A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}, \qquad \text{(b)} \quad A = \begin{pmatrix} 1 & -3 & 7 \\ 0 & 4 & 5 \\ 0 & 1 & 2 \end{pmatrix}.$$

Solution (a). Because $\det(A) = 8 - 5 = 3$, we see that

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix}.$$

Solution (b). By row reduction

$$\begin{aligned}
\begin{pmatrix} A & | & I \end{pmatrix} &= \left(\begin{array}{ccc|ccc} 1 & -3 & 7 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 4 & 5 & 0 & 1 & 0 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 13 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 & 1 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 13 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & \frac{4}{3} \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{13}{3} & -\frac{43}{3} \\ 0 & 1 & 0 & 0 & \frac{2}{3} & -\frac{5}{3} \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & \frac{4}{3} \end{array}\right).
\end{aligned}$$

Therefore

$$A^{-1} = \begin{pmatrix} 1 & \frac{13}{3} & -\frac{43}{3} \\ 0 & \frac{2}{3} & -\frac{5}{3} \\ 0 & -\frac{1}{3} & \frac{4}{3} \end{pmatrix}.$$

Alternative Solution (b). Because A can be partitioned as

$$A = \begin{pmatrix} 1 & \mathbf{c}^T \\ \mathbf{0} & B \end{pmatrix}, \qquad \text{where} \quad \mathbf{c} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix},$$

we seek A^{-1} in the form

$$A^{-1} = \begin{pmatrix} 1 & \mathbf{d}^T \\ \mathbf{0} & B^{-1} \end{pmatrix}.$$

Then

$$I = AA^{-1} = \begin{pmatrix} 1 & \mathbf{c}^T \\ \mathbf{0} & B \end{pmatrix}\begin{pmatrix} 1 & \mathbf{d}^T \\ \mathbf{0} & B^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{d}^T + \mathbf{c}^T B^{-1} \\ \mathbf{0} & I \end{pmatrix},$$

which implies that $\mathbf{d}^T = -\mathbf{c}^T B^{-1}$. By part (a) we have

$$B^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix}, \qquad \mathbf{c}^T B^{-1} = \frac{1}{3}\begin{pmatrix} -3 & 7 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -13 & 43 \end{pmatrix},$$

whereby

$$A^{-1} = \begin{pmatrix} 1 & -\mathbf{c}^T B^{-1} \\ \mathbf{0} & B^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{13}{3} & -\frac{43}{3} \\ 0 & \frac{2}{3} & -\frac{5}{3} \\ 0 & -\frac{1}{3} & \frac{4}{3} \end{pmatrix}.$$