Quiz 11 Solutions, Math 246, Professor David Levermore
Tuesday, 1 December 2009

(1) [5] The 2×2 matrix $\mathbf{A}$ has the real eigenpairs $\left(1\,,\begin{pmatrix}2\\1\end{pmatrix}\right)$ and $\left(3\,,\begin{pmatrix}-1\\2\end{pmatrix}\right)$.

(a) Sketch a phase portrait for the system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$. Indicate typical trajectories.

(b) For each portrait identify its type and state whether the origin is attracting, stable, unstable, or repelling.

Solution: Because the eigenvalues of $\mathbf{A}$ are real and of have the same sign, the phase portrait is a *nodal source*. Because all trajectories move away from the origin as t increases, it is *repelling* (and therefore also *unstable*).

- Your phase portrait should indicate trajectories on the line $c_1\begin{pmatrix}2\\1\end{pmatrix}$ move away from the origin with arrows.
- It should indicate trajectories on the line $c_2\begin{pmatrix}-1\\2\end{pmatrix}$ move away from the origin with double arrows.
- It should also indicate a family of trajectories move away from the origin along curves that emerge from the origin tangent to the line $c_1\begin{pmatrix}2\\1\end{pmatrix}$ and further away become more parallel to the line $c_2\begin{pmatrix}-1\\2\end{pmatrix}$.

(2) [5] Let $\mathbf{A} = \begin{pmatrix}-2 & 5\\ -2 & 0\end{pmatrix}\mathbf{x}\,.$

(a) Sketch a phase portrait for the system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$. Indicate typical trajectories.

(b) For each portrait identify its type and state whether the origin is attracting, stable, unstable, or repelling.

Solution: The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 2z + 10 = (z+1)^2 + 3^3\,.$$

The eigenvalues of $\mathbf{A}$ are therefore $-1+i3$ and $-1-i3$, whereby the phase portrait is a *spiral sink*. Because $a_{21} = -2 < 0$ the spiral will be *clockwise*. Because all trajectories move towards the origin as t increases, it is *attracting* (and therefore also *stable*).

Your phase portrait should indicate a family of trajectories that spiral into the origin in a clockwise fashion.