Kevin Smith – Matlab Second Order Equation.

The following phase portraits depict the behavior of the second order differential equation \((D2x/D2t)+b(dx/dt)^3-x^2+x=0\). The sequence of pictures are a result of changing the variable "b" from 0-1, and even beyond. The behavior of the function changes drastically immediately after "b" becomes non-zero. The stability of the origin becomes immediately affected however it remains stable. The saddle point at (1,0) that originally existed when b=0 has now drastically changed to create an almost inverted version of the previous picture. As "b" goes to infinity these points become increasingly less stable until nearly becoming unstable. One can see this in the last picture where "b" is increased to 5 and then 100. The stationary point (1,0) has now been taken over and (0,0) has now taken the shape of an unstable saddle point. However, it is interesting to note that even though these points appear to become unstable, the linearization (as shown in the handwritten derivations) shows that even as "b" becomes non-negative the matrices of the stationary points stay the same because "y" is still 0. Therefore, the stationary points (0,0) and (1,0) should remain as a stable center and an unstable saddle point. The shape of the graph outside of the stationary points drastically changes after “b” becomes non-zero because of the y dependent cubic function that now exists. The tangent lines which the level curves approach as “t” decreases are no longer linear because the derivative of the function “y^3”
becomes “$y^2$.” This is clearly seen as the tangent lines are parabolic arcs rather than straight lines when “$b$” is non-zero.
\[ \frac{dx}{dt}^2 + b \left( \frac{dx}{dt} \right)^3 - x^2 + x = 0 \]

\[ f(x,y) = \frac{dx}{dt} = y \]

\[ g(x,y) = \frac{dy}{dt} = -b \left( \frac{dx}{dt} \right)^3 + x^2 - x = -b \left( y^3 \right) + x^2 - x \]

**Level Curves**

\[ \frac{dy}{dx} = \frac{-b y^2 + x^2 - x}{y} \]

\[ H(x,y) = \frac{y^2 + b y^3 x + x^2 - x^3}{2} \]

**Stationary Points**

0 = y

0 = b(0)^3 + x^2 - x

\( x(1-x) = 0 \)

**Linearization**

\[
\begin{pmatrix}
\frac{dy}{dx} & \frac{dy}{dt} \\
\frac{dx}{dt} & \frac{dx}{dt}
\end{pmatrix}
= \begin{pmatrix} 0 & 1 \\ 2x - 1 & 3y^2(b) \end{pmatrix}
\]

at stationary point

\( (0, 0) \)

\[ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

\[ p(z) = 2 - z \quad p(z) = z^2 + 1 \]

roots \( (z+1)(z-1) \)

\( \lambda = -1, 1 \)

saddle, unstable

repelling

eigen vector \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)
\begin{align*}
b =& 0 \\
f &= @ (x,y) (y \cdot y^2/2 + b \cdot (y \cdot y^3) + x \cdot x^2/2 - (x \cdot x^3)/3; \\
Z &= f(X,Y); \\
level &= -.5:.005:.5; \\
contour(X,Y,Z,\text{level}) \\
axis([-2 2 -2 2]) \\
grid on
\end{align*}
b = 0.1

f = @(x,y)((y.^2)/2) + (b*(y.^3)*x) + (x.^2)/2 - (x.^3)/3;

Z = f(X,Y);

level = -3:.009:3;

contour(X,Y,Z,level)

axis([-2 2 -1 1])

grid on
b = 0.2

f=@(x,y)((y.^2)/2)+(b*(y.^3)*x)+(x.^2)/2 -(x.^3)/3;
Z=f(X,Y);
level=-.3:.009:.3;
contour(X,Y,Z,level)
axis([-2 2 -1 1])
grid on

b =

0.2000
b=3

\[ f(x,y) = \frac{(y^2)}{2} + (b(y^3)x) + \frac{(x^2)}{2} - \frac{(x^3)}{3}; \]

\[ Z=f(X,Y); \]

level=-3:.009:3;

contour(X,Y,Z,level)

axis([-2 2 -1 1])

grid on

\[ b = \]

0.3000
\[ f(x,y) = \left( \frac{y^2}{2} \right) + \left( b \times \frac{y^3}{3} \right) \times x + \left( \frac{x^2}{2} \right) - \left( \frac{x^3}{3} \right) \]

\[ Z = f(X,Y) \]

\[ \text{level} = -3:0.009:3; \]

\[ \text{contour}(X,Y,Z,\text{level}) \]

\[ \text{axis([-2 2 -1 1])} \]

\[ \text{grid on} \]
```matlab
b = 0.5
f = @(x,y)(y.^2/2)+(b*(y.^3)*x)+(x.^2)/2 - (x.^3)/3;
Z = f(X,Y);
level = -3:.009:.3;
contour(X,Y,Z,level)
axis([-2 2 -1 1])
grid on
```

```
b =
```

```
0.5000
```
b=.6

f=@(x,y)((y.^2)/2)+(b*(y.^3)*x)+(x.^2)/2 - (x.^3)/3;

Z=f(X,Y);

level=-.3:.009:.3;

contour(X,Y,Z,level)

axis([-2 2 -1 1])

grid on

b =

0.6000
b=.7

f=@(x,y)((y.^2)/2)+(b*(y.^3)*x)+(x.^2)/2 -(x.^3)/3;

Z=f(X,Y);

level=-3:.009:3;

contour(X,Y,Z,level)

axis([-2 2 -1 1])

grid on

b =

0.7000
b = 0.8

f=@(x,y)((y.^2)/2)+(b*(y.^3)*x)+(x.^2)/2 - (x.^3)/3;

Z = f(X,Y);

level = -.3:.009:.3;

contour(X,Y,Z,level)

axis([-2 2 -1 1])

grid on
\[ b = .9 \]
\[ f = @(x,y)((y.^2)/2)+(b*(y.^3)*x)+(x.^2)/2 - (x.^3)/3; \]
\[ Z = f(X,Y); \]
\[ \text{level} = -.3:.009:.3; \]
\[ \text{contour}(X,Y,Z,\text{level}) \]
\[ \text{axis([-2 2 -1 1])} \]
\[ \text{grid on} \]

\[ b = 0.9000 \]
b=1

f=@(x,y)((y.^2)/2)+(b*(y.^3)*x)+(x.^2)/2 -(x.^3)/3;

Z=f(X,Y);

level=-3:.009:3;

contour(X,Y,Z,level)

axis([-2 2 -1 1])

grid on
b=5

\[ f(x,y) = \frac{y^2}{2} + b \left( \frac{y^3}{3} \right)x + \frac{x^2}{2} - \frac{x^3}{3}; \]

\[ Z = f(X,Y); \]

\[ \text{level} = -.3:.009:.3; \]

\[ \text{contour}(X,Y,Z,\text{level}) \]

\[ \text{axis}([-6 1.5 -0.4 0.4]) \]

\[ \text{grid on} \]
\begin{verbatim}
\texttt{b=100}
\texttt{f=@(x,y)((y.^2)/2)+(b*(y.^3)*x)+(x.^2)/2 - (x.^3)/3;}
\texttt{Z=f(X,Y);}
\texttt{level=-3:.009:3;}
\texttt{contour(X,Y,Z,level)}
\texttt{axis([-5 1.5 -7.7])}
\texttt{grid on}
\end{verbatim}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{plot.png}
\end{figure}