Quiz 10 Solutions, Math 246, Professor David Levermore
Wednesday, 30 April 2008

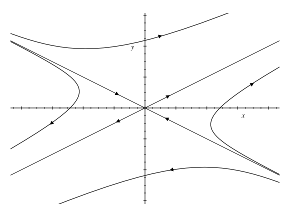
(1) [4] $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 4 & -1 \end{pmatrix}$ has eigenvalues -3 and 7. Find an eigenvector for each eigenvalue.

Solution: One has $\mathbf{A}+3\mathbf{I} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$ and $\mathbf{A}-7\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}$. The eigenvectors $\mathbf{v}_1$ associated with -3 satisfy $(\mathbf{A}+3\mathbf{I})\mathbf{v}_1 = 0$. These have the form $\mathbf{v}_1 = \alpha_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for every $\alpha_1 \neq 0$. The eigenvectors $\mathbf{v}_2$ associated with 7 satisfy $(\mathbf{A} - 7\mathbf{I})\mathbf{v}_2 = 0$. These have the form $\mathbf{v}_2 = \alpha_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for every $\alpha_2 \neq 0$.

(2) [3] The 2×2 matrix $\mathbf{A}$ has the real eigenpairs $\left(-1, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$ and $\left(3, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$.

Sketch a phase portrait for the system $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$. Indicate typical trajectories.

Solution: Because the eigenvalues of $\mathbf{A}$ are real and of opposite sign, the phase portrait is a *saddle*. Trajectories on the line $c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ move towards the origin while those on the line $c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ move away from the origin. The phase portrait is therefore



(3) [3] Sketch a phase portrait for the system $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$.

Solution: The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 2z + 2 = (z+1)^2 + 1 .$$

The eigenvalues of $\mathbf{A}$ are therefore $-1+i$ and $-1-i$, whereby the phase portrait is a *spiral sink*. Because

$$\text{for} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{one has} \quad \frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix},$$

so the spiral will be counterclockwise. One can use any point $\mathbf{x}$ to reach the same conclusion. The phase portrait is therefore