Quiz 7 Solutions, Math 246, Professor David Levermore
Wednesday, 2 April 2008

Short Table: $\mathcal{L}[\sin(bt)](s) = \dfrac{b}{s^2+b^2}$ for $s > 0$, $\quad \mathcal{L}[e^{at}](s) = \dfrac{1}{s-a}$ for $s > a$.

(1) [3] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for $f(t) = u(t-3)$, where u is the unit step function.

Solution: By the definitions of the Laplace transform and the unit step function

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t = \lim_{T\to\infty} \int_3^T e^{-st}\,\mathrm{d}t\,.$$

The above limit diverges for $s \leq 0$. For $s > 0$

$$\int_3^T e^{-st}\,\mathrm{d}t = -\frac{e^{-st}}{s}\bigg|_3^T = \frac{e^{-3s}}{s} - \frac{e^{-sT}}{s}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[\frac{e^{-3s}}{s} - \frac{e^{-sT}}{s}\right] = \frac{e^{-3s}}{s} \quad \text{for } s > 0\,.$$

(2) [4] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y = \sin(3t)\,, \qquad y(0) = y'(0) = 0\,. \qquad \text{DO NOT solve for } y(t)\text{, just } Y(s)\,!$$

Solution: The Laplace transform of the initial-value problem and item 1 in the table at the top of the page gives

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = \mathcal{L}[\sin(3t)](s) = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}\,,$$

where

$$\begin{aligned}
\mathcal{L}[y](s) &= Y(s)\,,\\
\mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s)\,,\\
\mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s)\,.
\end{aligned}$$

Hence,

$$(s^2+4)Y(s) = \frac{3}{s^2+9}\,, \qquad \Longrightarrow \qquad Y(s) = \frac{3}{(s^2+4)(s^2+9)}\,.$$

(3) [3] Find the inverse Laplace transform of $F(s) = \dfrac{5s}{s^2+s-6}$.

Solution: Partial fractions and item 2 in the table at the top of the page gives

$$\begin{aligned}
F(s) &= \frac{5s}{s^2+s-6} = \frac{5s}{(s-2)(s+3)} = \frac{2}{s-2} + \frac{3}{s+3}\\
&= 2\mathcal{L}[e^{2t}](s) + 3\mathcal{L}[e^{-3t}](s) = \mathcal{L}\big[2e^{2t} + 3e^{-3t}\big](s)\,.
\end{aligned}$$

Therefore

$$\mathcal{L}^{-1}[F](t) = 2e^{2t} + 3e^{-3t}\,.$$